Concurrent Feedback/Feedforward Design for Second Order Systems

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***Abstract*— This paper studies the problem of concurrent design of a feedback controller and a pre-filter to minimize a weighted cost comprising the maneuver time and the input power. The pre-filter is parameterized as a time-delay filter motivated by the Posicast controller. The proposed technique is illustrated on a double integrator assuming that the feedback controller is of the proportional derivative form. A closed form expression relating the weighting parameter and the number of delays, to the feedback gains is derived. The proposed technique is also illustrated on spring-mass and spring-mass-dashpot systems.**

I. Introduction

Otto Smith introduced the concept of *Posicast* controller [1] where a step input is broken in multiple steps so as to eliminate residual oscillations of an under-damped system. It was clear that the *Posicast* controller is sensitive to errors in model parameters such as the natural frequency and damping ratio. Singer and Seering [2] proposed a technique to design a series of impulses which resulted in a response which was robust to errors in the system parameters. They referred to the impulse sequence as an *Input Shaper*. Singh and Vadali [8] illustrated that a time-delay filter designed to cancel the under-damped poles of the system resulted in the same solution as the input shaper. They also illustrated that cascading time-delay filters designed to cancel the uncertain under-damped poles resulted in a robust prefilter. There have been numerous other papers which have studied the problem of design of input shapers for rest-to-rest maneuvers of vibratory systems with the objective of minimizing the residual vibrations in the presence of modelling uncertainties [4], [5], [6], [7], [10], [11].

The input shapers and the time-delay filters assume that the system being studied is stable. The effect of the feedback controller on the prefilter has recently been studied by Kenison and Singhose [3], Muenchhof and Singh [13], Tenne and Singh [12] etc. Kenison and Singhose [3] study the problem of simultaneous design of a PD controller and an Input shaper for a double integrator. They consider various metrics such as percent overshoot, residual energy, settling time etc. in their design. They conclude that the concurrent design results in better performance compared to a PD controller. Muenchhof and Singh [13] study the problem of simultaneous design of feedback/feedforward controller to minimize the cost function which include the residual energy in the system and a metric which represents the disturbance rejection capability of the closed loop system. The problem is formulated as a minimax controller which minimize the maximum cost over a hypercube of uncertainties. They illustrated their controller on the benchmark floating oscillator problem. The minimax design problem is computationally expensive and to address this issue, Tenne and Singh [12], proposed to exploit the *Unscented Transformation* to represent the distribution of the cost function as a function of the distribution of the uncertain model parameters. They solved the minimax problem where the cost is a combination of the residual energy, maneuver time and the integral of a quadratic cost of the states and control. This concurrent design was carried out to design a feedback controller and a prefilter for a high-speed tape transport system.

In this work, the simultaneous design of feedback/feedforward controller is presented for a set of second order systems. The double integrator is considered in Section II and is followed by concurrent design of feedback/feedforward controllers for spring-mass-dashpot systems in Section III. The paper concludes with a summary of results in Section IV.

II. Double Integrator

Consider the block diagram shown in Figure 1 which represented the feedback control of a double integrator. The feedback controller is parameterized as a proportional-derivative controller since it guarantees stability provided both the gains are positive. The reference input to the system is modified by a time-delay filter. The problem considered here is the rest-to-rest maneuver of the system.

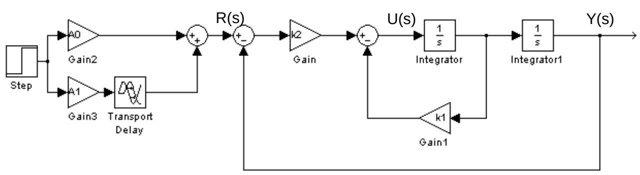

Fig. 1. Closed Loop System

The objective of the concurrent design is to arrive at the parameters k_1, k_2, A_0, A_1 and T so as to minimize the cost function:

$$J = \int_0^T (1 + \alpha u^2)\, dt. \tag{1}$$

The transfer function of the closed loop system is

$$\frac{Y(s)}{R(s)} = \frac{k_2}{s^2 + k_1 s + k_2} \tag{2}$$

and the transfer function relating the reference input to the control input is:

$$\frac{U(s)}{R(s)} = \frac{k_2 s^2}{s^2 + k_1 s + k_2} \tag{3}$$

where $R(s)$ is the Laplace transform of the prefiltered step input. Including the transfer function of a single time-delay filter, the control input $U(s)$ can be represented as:

$$U(s) = \frac{(A_0 + A_1 exp(-sT))k_2 s}{s^2 + k_1 s + k_2}. \tag{4}$$

For time $t < T$, the control input $u(t)$ can be solved in closed form as:

$$u(t) = A_0 k_2 exp(-0.5k_1 t)\left(cos(\sqrt{4k_2 - k_1^2}t/2) - \frac{k_1 sin(\sqrt{4k_2 - k_1^2}t/2)}{\sqrt{4k_2 - k_1^2}}\right). \tag{5}$$

Defining $a = 0.5k_1$ and $b = \sqrt{4k_2 - k_1^2}/2$, the control input in time-domain can be represented as:

$$u(t) = A_0 k_2 e^{-at}\left(cos(bt) - \frac{asin(bt)}{b}\right) \tag{6}$$

and $u^2(t)$ is:

$$\begin{aligned} u^2(t) &= A_0^2 k_2^2 e^{-2at}\left(\frac{a^2 sin^2(bt) + b^2 cos(bt) - 2ab\, sin(bt)cos(bt)}{b^2}\right) \\ &= A_0^2 k_2^2 e^{-2at}\left(\frac{(a^2+b^2) + (b^2-a^2)cos(2bt) - 2ab\, sin(2bt)}{2b^2}\right) \\ &= A_0^2 k_2^2 e^{-2at}(P + Qcos(2bt) + Rsin(2bt)). \end{aligned} \tag{7}$$

To determine the integral of $u^2(t)$, consider the term:

$$\frac{1}{A_0^2 k_2^2}\int_0^T u^2(t)dt = \int_0^T e^{-2at}(P + Qcos(2bt) + Rsin(2bt))dt \tag{8}$$

which can be shown to reduce to

$$\int_0^T u^2(t)\, dt = A_0^2 k_2^2 \frac{(1 - e^{-k_1 T})}{2k_1} \tag{9}$$

Thus, the cost function can now be represented in closed form as

$$J = T + \frac{A_0^2 k_2^2 \alpha (1 - e^{-k_1 T})}{2k_1} \tag{10}$$

The goal of time-delay filters to eliminate residual vibrations, is achieved by requiring a pair of zeros of the time-delay filter to cancel the poles of the closed loop system. The closed loop poles of the system are located at:

$$s = -\frac{k_1}{2} \pm \frac{\sqrt{k_1^2 - 4k_2}}{2} = -\frac{k_1}{2} \pm j\frac{\sqrt{4k_2 - k_1^2}}{2}. \tag{11}$$

Now, the parameters of the time-delay filter can be represented in closed form as:

$$T = \frac{2\pi}{\sqrt{4k_2 - k_1^2}}, \tag{12}$$

$$A_0 = \frac{e^{0.5k_1T}}{1 + e^{0.5k_1T}} \tag{13}$$

$$A_1 = \frac{1}{1 + e^{0.5k_1T}} \tag{14}$$

which satisfies the requirement that $A_0 + A_1 = 1$. The cost can now be simplified to

$$J = T + \frac{k_2^2\alpha}{2k_1}\frac{(e^{0.5k_1T} - 1)}{(e^{0.5k_1T} + 1)}. \tag{15}$$

To derive the optimal feedback gains, we need to determine the gradients of the cost function to the variables k_1 and k_2 which result in the equations:

$$\begin{aligned}\frac{\partial J}{\partial k_1} =& \frac{\partial T}{\partial k_1} - \frac{k_2^2\alpha}{k_1^2}\frac{(e^{0.5k_1T} - 1)}{(e^{0.5k_1T} + 1)} \\ &+ \frac{k_2^2\alpha}{2k_1}\frac{e^{0.5k_1T}}{(e^{0.5k_1T} + 1)^2}(k_1\frac{\partial T}{\partial k_1} + T).\end{aligned} \tag{16}$$

It can be shown that the gradient is zero in the limit for any value of k_2:

$$\lim_{k_1 \to 0}\frac{\partial J}{\partial k_1} = 0. \tag{17}$$

The gradient of J with respect to k_2 results in the equation

$$\begin{aligned}\frac{\partial J}{\partial k_2} =& \frac{\partial T}{\partial k_2} + \frac{k_2\alpha}{k_1}\frac{(e^{0.5k_1T} - 1)}{(e^{0.5k_1T} + 1)} \\ &+ \frac{k_2^2\alpha}{2k_1}\frac{e^{0.5k_1T}}{(e^{0.5k_1T} + 1)^2}(k_1\frac{\partial T}{\partial k_2})\end{aligned} \tag{18}$$

It can now be shown that:

$$\begin{aligned}\lim_{k_1 \to 0}\frac{\partial J}{\partial k_2} &= \frac{-\pi}{2\sqrt{k_2^3}}(1 + \frac{k_2^2\alpha}{8}) + \frac{\pi\sqrt{k_2}\alpha}{4} = 0 \\ \Rightarrow k_2 &= \sqrt{\frac{8}{3\alpha}}\end{aligned} \tag{19}$$

which states that as the penalty on the power increases, the proportional feedback gain decreases. Figure 2 illustrates the variation of the cost function as a function of k_1 and k_2, for $\alpha = 1$. It can be seen that the analytically derived solution matches the graphical minimum of the cost function.

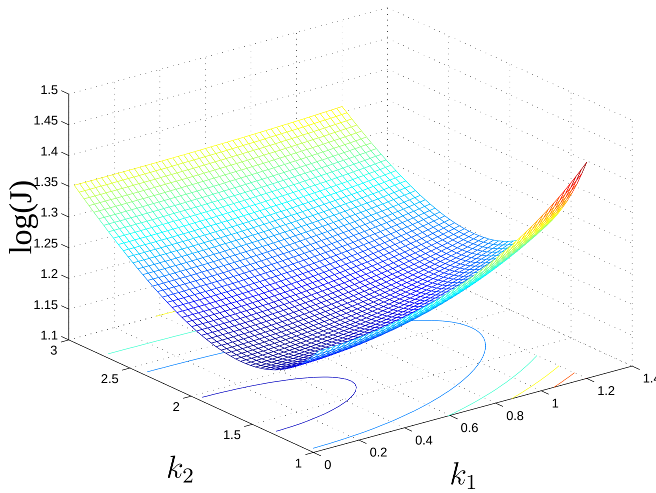

Fig. 2. Cost Function

Singh and Vadali [9] illustrated that by placing multiple zeros of the time-delay filter at the expected location of the under-damped poles of the system resulted in robustness to uncertainties in damping ratio and natural frequencies. To study the improvement in the robustness of the two time-delay prefilter, the concurrent design problem is studied for the two time-delay filter. The control input for a two time-delay prefilter is:

$$U(s) = \frac{(B_0 + B_1exp(-sT) + B_2exp(-2sT))k_2s}{s^2 + k_1s + k_2}. \tag{20}$$

where the transfer function of the two time-delay filter is:

$$\begin{aligned}G(s) &= B_0 + B_1exp(-sT) + B_2exp(-2sT) \\ &= (A_0 + A_1exp(-sT))^2\end{aligned} \tag{21}$$

where the gains A_0 and A_1 are given by Equations 13 and 14. This corresponds to locating multiple zeros of the time-delay filter at the nominal location of the closed loop poles of the system. The gains of the time-delay filter generated by cascading multiple single time-delay filters in series results in a time-delay filter where all the gains are positive and less than 1. Furthermore, the time-delays are integral multiples of the period of the damped natural frequency of the closed loop system. Following the same procedure as the single time-delay filter, it can be shown that the optimal feedback gains are:

$$k_1 = 0, \quad k_2 = 2\sqrt{\frac{8}{3\alpha}} \tag{22}$$

where the proportional feedback gain is double that of the single time-delay filtered problem. Increasing the number of delays in the time-delay filter, the optimal feedback gains can be solved and the resulting derivative feedback gain is zero. Assuming that the derivative feedback gain is zero irrespective of the number of delays in the time-delay filter, a closed form solution for the optimal proportional feedback gain can be derived.

Let n be the number of time-delays. For $k_1 = 0$, the value of optimal k_2 in closed form can be obtained as follows. The cost function for n delays is:

$$J = \int_0^{nT} \alpha u^2 \, dt + nT. \tag{23}$$

where,

$$T = \frac{\pi}{\sqrt{k_2}}.$$

This can be rewritten as

$$J = \alpha \left(\int_0^T u_0^2 \, dt + \int_T^{2T} u_1^2 \, dt + ... + \int_{(n-1)T}^{nT} u_{n-1}^2 \, dt \right) + nT. \tag{24}$$

where,

$$u_{n-1} = A_0 k_2 \cos(\sqrt{k_2} t) + A_1 k_2 \cos(\sqrt{k_2}(t-T)) + ... + A_{n-1} k_2 \cos(\sqrt{k_2}(t-(n-1)T)). \tag{25}$$

$$A_i = \frac{{}^nC_i}{2^n}. \tag{26}$$

It can be shown that,

$$\int_{iT}^{(i+1)T} u_i^2 \, dt = \frac{k_2^2 T}{2} (A_0 - A_1 + A_2 ... + (-1)^i A_i)^2. \tag{27}$$

Substituting A_i into Equation 24, results in the equation

$$J = \frac{\alpha k_2^2 T}{2^{2n+1}} \left((C_0)^2 + (C_0 - C_1)^2 + ... + (C_0 - C_1 + C_2 - ... + (-1)^{n-1} C_{n-1})^2 \right) + nT. \tag{28}$$

where in the interest of brevity ${}^nC_i$ is represented as C_i. Substituting for T and simplifying, this further reduces to

$$J = \frac{\alpha k_2^2 \pi ({}^{2n-2}C_{n-1})}{2^{2n+1}\sqrt{k_2}} + \frac{n\pi}{\sqrt{k_2}}. \tag{29}$$

Finding $\frac{dJ}{dk_2}$ and equating to zero, we can calculate the optimal k_2 as

$$k_2 = \sqrt{\frac{n 2^{2n+1}}{3\alpha({}^{2n-2}C_{n-1})}}. \tag{30}$$

The optimal feedback gains of the proportional derivative controller are presented in Table I. It can be seen that with the increase in the number of delays, the optimal proportional feedback gain increases. However, since the gains of the time-delay filter decrease with an increase in the number of delays, the peak magnitude of the control profile does not change significantly as shown in Figure 5.

# of Delays	k_1	k_2
1	0	$\sqrt{\frac{8}{3\alpha}}$
2	0	$2\sqrt{\frac{8}{3\alpha}}$
3	0	$\sqrt{\frac{64}{3\alpha}}$
4	0	$\sqrt{\frac{512}{15\alpha}}$
5	0	$\sqrt{\frac{1024}{21\alpha}}$
6	0	$\sqrt{\frac{4096}{63\alpha}}$
n	0	$\sqrt{\frac{n2^{2n+1}}{3\alpha({}^{2n-2}C_{n-1})}}$

TABLE I

OPTIMAL FEEDBACK GAINS

Figure 3 and 4 illustrates the displacement and velocity response of the double integrator for the optimal feedback/feedforward controller for a unit step input. The obvious penalty is the increase in the maneuver time which brings with it robustness to modelling uncertainties. Figure 5 illustrates the control profile and with an increase in the number of delays there is an increase in the number of serrations. Since the time-delay prefilter converts the unit step into a staircase form, the jump discontinuities in the control profile correspond to the steps in the prefiltered reference profile.

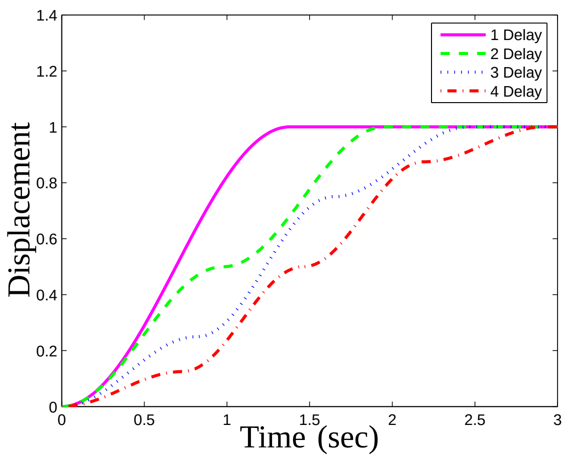

Fig. 3. Displacement Response

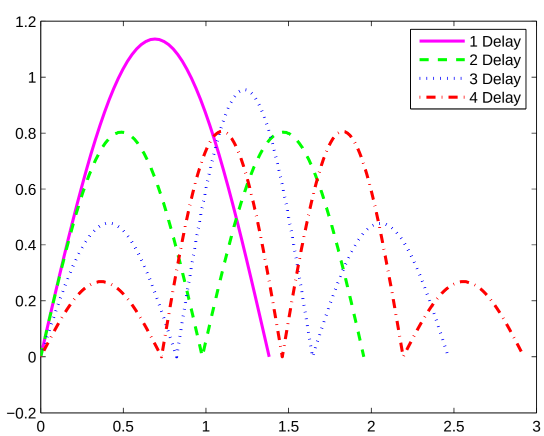

Fig. 4. Velocity Response

The maximum magnitude of the control profile can be determined from Equation 25 which results in the solution

$$u_{max} = \frac{{}^{n-1}C_i}{2^n}\sqrt{\frac{n2^{2n+1}}{3\alpha({}^{2n-2}C_{n-1})}}. \tag{31}$$

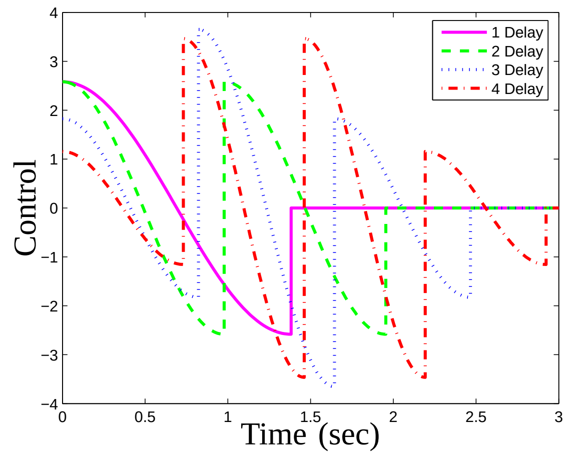

Fig. 5. Control Input

where

$$i = \begin{cases} \frac{n+1}{2} & \text{if } \ n \text{ is odd} \\ \frac{n}{2} & \text{if } \ n \text{ is even.} \end{cases} \tag{32}$$

Figure 6 illustrates the variation of the maximum magnitude of the control profile as a function of the weighting parameter α, which is monotonically decreasing. This graph can be exploited to select α given constraint on the control magnitude.

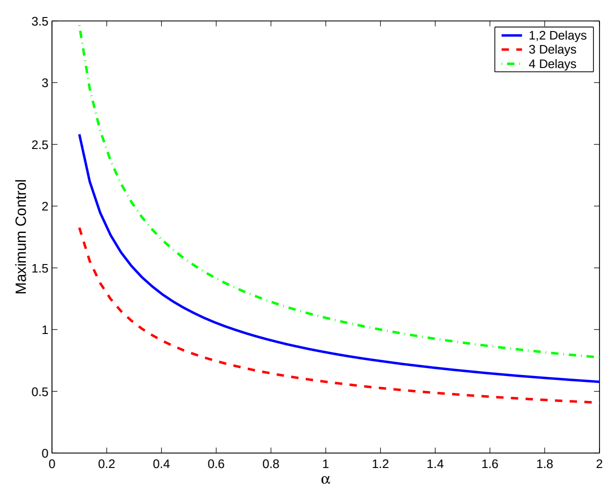

Fig. 6. Variation of Peak Control

The motivation for adding delays to the time-delay filter was to decrease the sensitivity of the residual energy to errors in estimated inertia of the system. To illustrate the benefits of additional delays, the variation of the residual energy as a function of varying inertia of the system is plotted in Figure 7. The solid, dashed, dotted and dash-dot lines correspond to prefilters with 1,2,3 and 4 time-delays respectively. It is patent from Figure 7 that with the increase in the number of delays, the insensitivity of the residual energy is increased.

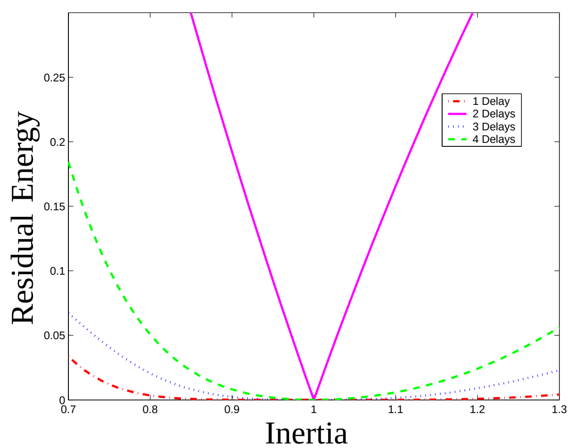

Fig. 7. Sensitivity

Figure 8 illustrates the time evolution of the displacement of the double integrator with a 100% error in estimated inertia. It is clear that the dash-dot line which corresponds to the 4 time-delay prefilter has the smallest residual energy.

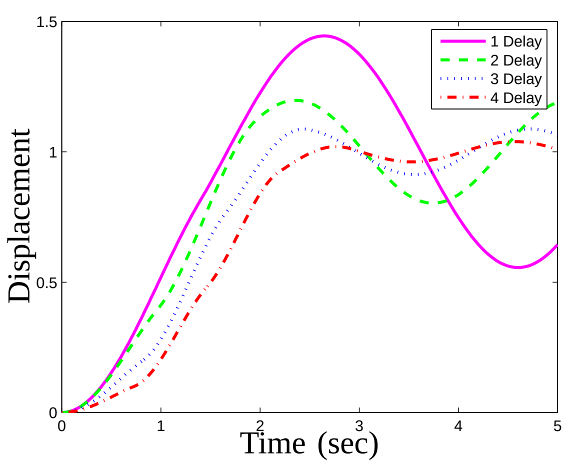

Fig. 8. Displacement

III. Mass/Spring/Damper System

This section illustrates the generalization of the concurrent design techniques on a general second order system.

A. Spring-Mass-Damper System

Consider the second order system:

$$\ddot{y} + c\dot{y} + ky = u \tag{33}$$

subject to the control law:

$$u = -k_2(y - r) - k_1\dot{y} \tag{34}$$

The transfer function of the system is

$$\frac{Y(s)}{R(s)} = \frac{k_2}{s^2 + (k_1 + c)s + (k_2 + k)} \tag{35}$$

and

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + cs + k}$$

Pre-filtering the step input to the system with a single time-delay filter results in:

$$U(s) = \frac{(k_2(s^2 + cs + k))(A_0 + A_1 e^{-sT})}{s(s^2 + (k_1 + c)s + (k_2 + k))}$$

For time $t < T$ the control input $u(t)$ can be solved as

$$\begin{aligned} u(t) =& \frac{k_2 A_0 e^{-aT}}{k_2 + k}(k_2 cos(bt) \\ &+ \frac{(ck_2 - k_1k_2 + 2k_1k)sin(bt)}{2b}) + \frac{A_0 k_2 k}{k_2 + k} \end{aligned} \tag{36}$$

where

$$\begin{aligned} a &= -\frac{k_1 + c}{2} \\ b &= \frac{\sqrt{4(k_2 + k) - (k_1 + c)^2}}{2} \\ T &= \frac{\pi}{b} \\ A_0 &= \frac{e^{aT}}{1 + e^{aT}} \end{aligned}$$

and

$$u_{ss} = \frac{A_0 k_2 k}{k_2 + k}$$

It is clear that the steady state output of the system to a unit step input is scaled by a factor $\frac{k_2}{k_2+k}$ which results in a steady state error. Since, this is a type 0 system, the control input is non-zero at steady state which prompts us to propose the cost function:

$$J = \int_0^T \left(1 + \alpha(u - u_{ss})^2\right)\, dt$$

where u_{ss} is the steady state value of the control input u.

This cost function in closed form can be shown to be

$$J = T + \frac{\alpha k_2^2((k_1 k - c k_2)^2 + k_2^2(k_2 + k))(e^{\frac{(k_1+c)}{2}T} - 1)}{2(k_1+c)(k_2+k)^3(e^{\frac{(k_1+c)}{2}T} + 1)} \quad (37)$$

The optimal feedback gains and the parameters of the time-delay filter can be numerically determined. Figures 9 and 10 illustrate the variation of the optimal feedback gains as a function of the damping factor c and the stiffness k.

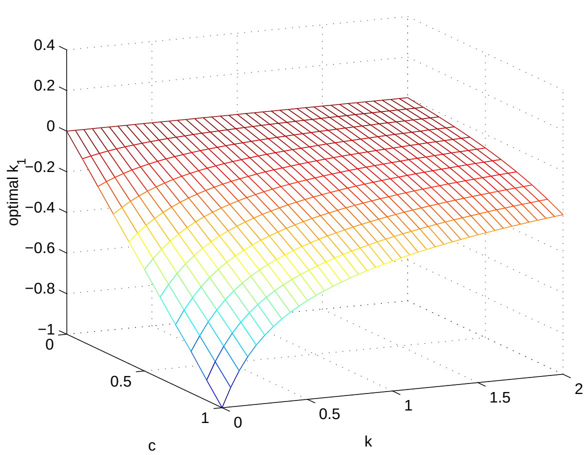

Fig. 9. Optimal k_1

Closed form solutions for the optimal gains as a function of c and k for a general second order system are indeterminable. However, closed form solutions can be found for mass-damper and spring-mass systems as shown in the following sections.

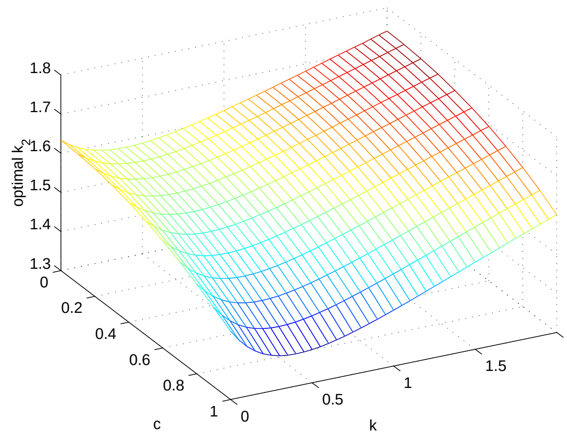

Fig. 10. Optimal k_2

B. Mass-Damper System

Letting $k = 0$ in Equation 37 gives

$$J = T + \frac{\alpha k_2^2(c^2 + k_2)(e^{\frac{k_1+c}{2}T} - 1)}{2(k_1+c)k_2(e^{\frac{k_1+c}{2}T} + 1)}$$

It can be shown that this cost is minimum when the gains are $k_1 = -c$ and k_2 is the solution of the equation

$$3\alpha k_2^2 + \alpha k_2 c^2 - 8 = 0$$

which is:

$$k_2 = \frac{-\alpha c^2 \pm \sqrt{\alpha^2 c^4 + 96\alpha}}{6\alpha}. \quad (38)$$

Since k_2 is required to be positive for stability, the optimal solution is unique.

The optimal derivative gain can be seen to force the closed loop dynamics to include undamped poles which is identical to the closed loop response of the double integrator. A conservative design when the damping coefficient c is uncertain is to select k_1 to equal the upper bound of c.

C. Spring-Mass System

Next, we consider a spring-mass system. Let $c = 0$ and $k = \omega^2$ in Equation 37. The cost function then becomes

$$J = T + \frac{\alpha k_2^2(k_1^2\omega^4 + k_2^2(k_2 + \omega^2)))(e^{\frac{k_1}{2}T} - 1)}{2k_1(k_2+\omega^2)^3(e^{\frac{k_1}{2}T} + 1)}$$

It can be shown that this cost is minimum when the gains are $k_1 = 0$ and k_2 is the solution to the equation

$$3\alpha k_2^4 + 8\alpha k_2^3\omega^2 - 8k_2^2 - 16k_2\omega^2 - 8\omega^4 = 0 \quad (39)$$

From Equation 35 it is clear that $k_2 + \omega^2$ is required to be positive for stability. Replacing k_2 with $x - \omega^2$ in Equation 39 we get

$$3\alpha x^4 - 4\alpha x^3\omega^2 - (8+6\alpha\omega^4)x^2 + 12\alpha\omega^6 x - 5\alpha\omega^8 = 0 \quad (40)$$

whose roots are $k_2 + \omega^2$. The discriminant of the quartic equation can be shown to be negative which implies that the quartic equation has two real and two complex roots [14]. Moreover, since the coefficient of x^0 in Equation 40 is negative, which corresponds to the product of the roots, this proves that there is only one real root such that $k_2 + \omega^2 > 0$ for Equation 39 thus guaranteeing an unique solution for optimal k_2

IV. Conclusions

Simultaneous design of a feedback controller in conjunction with a time-delay filter is studied in this work. A cost function which is a weighted combination of the maneuver time and the control power is minimized for a double integrator. It is assumed that the inertia of the double integrator is uncertain which results in uncertain location of the closed loop poles of the system. The feedback controller considered in this work is a proportional-derivative controller which is guaranteed to be stable for all positive proportional and derivative feedback gains. The time-delay filter is designed to cancel the nominal closed loop poles of the system. To desensitize the closed loop systems to uncertainties in the inertia, multiple zeros of the time-delay filters are located at the estimated closed loop poles. A closed form solution for the feedback gains is derived in terms of the number of delays and weighting parameter (α). It is shown that the proportional feedback gain increases with the number of delays and the derivative feedback gain is one which results in an undamped closed loop response. Numerical simulations illustrate the improvement in the insensitivity of the control system with an increase in the number of delays of the time-delay filter. The proposed technique is also illustrated on spring-mass-dashpot systems.

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