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# Polynomial Chaos Based Design of Robust Input Shapers

*A probabilistic approach, which exploits the domain and distribution of the uncertain model parameters, has been developed for the design of robust input shapers. Polynomial chaos expansions are used to approximate uncertain system states and cost functions in the stochastic space. Residual energy of the system is used as the cost function to design robust input shapers for precise rest-to-rest maneuvers. An optimization problem, which minimizes any moment or combination of moments of the distribution function of the residual energy is formulated. Numerical examples are used to illustrate the benefit of using the polynomial chaos based probabilistic approach for the determination of robust input shapers for uncertain linear systems. The solution of polynomial chaos based approach is compared with the minimax optimization based robust input shaper design approach, which emulates a Monte Carlo process.* [DOI: 10.1115/1.4001793]

## 1 Introduction

Precise point-to-point control is of interest in a variety of applications, including scanning probe microscopes, hard disk drives, gantry cranes, flexible arm robots, etc. All of these systems are characterized by underdamped modes, which are excited by the actuators. Precise position control mandates that the energy in the vibratory modes be dissipated by the end of the maneuver [1–4]. These are challenging demands on the controller when the model parameters are known precisely. However, uncertainties in model parameters are ubiquitous and these uncertainties manifest themselves as a deterioration in the performance of the controller. Input shaping is a technique, which shapes the reference command to the system so as to eliminate or minimize the residual energy [5]. The first solution to desensitize the input shaper to modeling uncertainties was to impose constraints on the sensitivity of the residual energy to model parameter errors to be zero at the end of the maneuver. The resulting solution is referred to as the zero vibration and derivative (ZVD) input shaper [1]. The ZVD input shaper improves the performance of the prefilter in the proximity of the nominal model of the system. To exploit knowledge of the domain of uncertainty, the extrainsensitive (EI) input shaper [6] and the minimax input shaper [7] were proposed, which are worst case designs, i.e., they minimize the worst performance of the system over the domain of uncertainty. The minimax input shaper requires sampling of the uncertain space, which results in a computationally expensive design as the dimension of the uncertain parameter space increases.

A technique to incorporate the distribution of uncertainty in the input shaper design process was proposed by Chang et al. [8], where the expected value of the residual vibration was minimized. The minimax input shaper [7] incorporated the probability distribution function as a weighting scheme to differentially weigh the plants that are sampled in the domain of uncertainty. Tenne and Singh [9] proposed the use of unscented transformation to map the Gaussian distributed uncertain parameters into the residual energy space. The unscented transformation force fits the distribution of the residual energy to a Gaussian. The sum of the mean and deviation of the residual energy distribution was minimized, which resulted in a robust input shaper. Clearly, this approach cannot represent non-Gaussian distributions, which limits its capability.

This paper formulates an optimization problem exploiting the strengths of polynomial chaos (PC) to determine representative parameters (moments or cumulants) of the probability density function of states of a linear dynamical system whose model parameters are random variables. The polynomial chaos expansion provides a computationally efficient approach compared with Monte Carlo (MC) simulation for the estimation of moments or cumulants of the functions of uncertain state variables. This paper is organized as follows: Following the introductory section, a brief overview of robust input shaper design is presented. This is followed by the development of the polynomial chaos based design of robust input shapers. The specific problems considered are a spring-mass system with an uncertain coefficient of stiffness and a three mass-spring system with uncertain mass and stiffness parameters. Numerical simulations are used to compare the results of robust minimax input shaper design with those of the polynomial chaos based design.

## 2 Input Shaping

Input shapers also referred to as time-delay filters are a simple and powerful approach for the shaping of reference input to eliminate or minimize residual motion of systems undergoing transition from one set point to another. The system being controlled is assumed to be stable or marginally stable and could represent an open-loop or a closed-loop system. One of the strengths of input shapers is their robustness to modeling uncertainties. The earliest solution to addressing the problem of sensitivity of the input shaper to uncertainties in model parameter errors involved forcing the sensitivity of the residual energy to model errors to zero at the end of the maneuver. This works well for perturbations of the parameters about the nominal values. In case one has knowledge of the domain of uncertainty and the distribution of the uncertain variable, this additional knowledge can be exploited in the design of an input shaper by posing a minimax optimization problem. In this formulation, the maximum magnitude of the residual energy over the domain of uncertainty is minimized. The minimax optimization problem is computationally expensive and as the dimension of the uncertain space grows, the *computational cost grows exponentially* [10] since the uncertain space has to be finely sampled.

Let us consider a linear mechanical system of the following form:

$$\mathbf{M}(\mathbf{p})\ddot{\mathbf{x}}(t,\mathbf{p}) + \mathbf{C}(\mathbf{p})\dot{\mathbf{x}}(t,\mathbf{p}) + \mathbf{K}(\mathbf{p})\mathbf{x}(t,\mathbf{p}) = \mathbf{D}(p)\mathbf{u}(t) \quad (1)$$

where $\mathbf{x}(t,\mathbf{p}) \in \mathbb{R}^n$ represents the generalized displacement coordinate with $\mathbf{u}(t) \in \mathbb{R}^m$ representing the deterministic system input vector. $\mathbf{M}(\mathbf{p})$, $\mathbf{C}(\mathbf{p})$, and $\mathbf{K}(\mathbf{p})$ are the mass, damping, and stiffness matrices, respectively. $\mathbf{p} \in \mathbb{R}^r$ is a vector of uncertain system parameters, which are functions of random variable $\boldsymbol{\xi}$ with known

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probability distribution function (pdf) $f(\boldsymbol{\xi})$, i.e., $\mathbf{p}=\mathbf{p}(\boldsymbol{\xi})$. For example, the uncertain variable $\mathbf{p}$ could be represented as

$$\mathbf{p} \sim \mathcal{U}[\mathbf{a},\mathbf{b}] \tag{2}$$

which implies that $\mathbf{p}$ is a uniformly distributed random variable and lies in the range

$$\mathbf{a} \leq \mathbf{p} \leq \mathbf{b} \tag{3}$$

or $\mathbf{p}$ could be a Gaussian variable

$$\mathbf{p} \sim \mathcal{N}[\boldsymbol{\mu},\boldsymbol{\Sigma}] \tag{4}$$

where $\boldsymbol{\mu}$ is the mean of the random variable $\mathbf{p}$ and $\boldsymbol{\Sigma}$ is the covariance matrix. $\mathbf{p}$ is not restricted to uniform or Gaussian distributions.

The transfer function of the input shaper/time-delay filter is parameterized as

$$G(s) = \sum_{i=0}^{Z} A_i e^{-sT_i} \tag{5}$$

where $T_0$ is zero and $Z$ is the number of delays in the prefilter. $A_i$ and $T_i$ are the parameters that need to be determined to satisfy the objective of completing the state transition with minimal excursion from the desired final states in the presence of modeling uncertainties.

For rest-to-rest maneuvers, the residual energy is a germane cost function. The residual energy at the final time $T_Z$ can be defined as

$$V(T_Z) = \frac{1}{2}\dot{\mathbf{x}}^T(T_Z)\mathbf{M}(\mathbf{p})\dot{\mathbf{x}}(T_Z) + \frac{1}{2}(\mathbf{x}(T_Z) - \mathbf{x}_f)^T\mathbf{K}(\mathbf{p})(\mathbf{x}(T_Z) - \mathbf{x}_f) \tag{6}$$

where $\mathbf{x}_f$ is the vector of the desired final displacement. The first term of $V(T_Z)$ is the kinetic energy in the system and the last term is the pseudopotential energy, which measures the energy resident in a hypothetical set of springs when $\mathbf{x}$ deviates from $\mathbf{x}_f$. If $\mathbf{K}(\mathbf{p})$ is not positive definite, the cost function $V(T_Z)$ has to be augmented with a quadratic term to ensure that the cost function is positive definite. The resulting cost function is

$$V(T_Z) = \frac{1}{2}\dot{\mathbf{x}}^T(T_Z)\mathbf{M}(\mathbf{p})\dot{\mathbf{x}}(T_Z) + \frac{1}{2}(\mathbf{x}(T_Z) - \mathbf{x}_f)^T\mathbf{K}(\mathbf{p})(\mathbf{x}(T_Z) - \mathbf{x}_f) + \frac{1}{2}(x_r - x_{rf})^2 \tag{7}$$

where $x_r$ refers to the rigid body displacement and $x_{rf}$ is the desired final position.

The design of a robust input shaper can be posed as the problem

$$\min_{A_i,T_i} \max_{\mathbf{p}} V(T_Z) \tag{8}$$

which is a minimax optimization problem and emulates a Monte Carlo process. This is also equivalent to minimizing the $\ell_\infty$ norm of the residual energy. The residual energy can be weighted by the distribution of the uncertain parameter [8,7], which results in a weighted minimax input shaper. This, as indicated earlier, is a computationally expensive problem when the dimension of $\mathbf{p}$ grows. To alleviate this problem, this paper endeavors to identify a probabilistic representation of the uncertain residual energy $V(T_Z)$ as a function of the uncertain parameter vector $\mathbf{p}(\boldsymbol{\xi})$. To emulate the minimax optimization problem in the probability space, one would require the distribution of $V(T_Z)$ to have a small mean and small compact support, which would correspond to small residual energy and small spread over the range of the random variable. In the next section, we describe a probabilistic method based on polynomial chaos, which exploits knowledge of the domain and distribution of the uncertain model parameters to calculate approximately the moments or cumulents of the residual energy.

## 3 Polynomial Chaos

Polynomial chaos is a term coined by Norbert Wiener in 1938 [11] to describe the members of the span of Hermite polynomial functionals of a Gaussian process. The basic idea of this approach is to approximate the stochastic system state in terms of finite-dimensional series expansion in the stochastic space. According to the Cameron–Martin theorem [12], such an expansion converges, in the $L^2$ sense, to any arbitrary stochastic process with finite variance (which applies to most physical processes). This approach is combined with the finite element method to model uncertainty in Ref. [13]. This has been generalized in Ref. [14] to efficiently use the different classes of orthogonal polynomials to model various probability distributions.

Let us consider a second order stochastic linear system given by Eq. (1).

$$\mathbf{M}(\mathbf{p})\ddot{\mathbf{x}}(t,\mathbf{p}) + \mathbf{C}(\mathbf{p})\dot{\mathbf{x}}(t,\mathbf{p}) + \mathbf{K}(\mathbf{p})\mathbf{x}(t,\mathbf{p}) = \mathbf{D}(\mathbf{p})\mathbf{u}(t) \tag{9}$$

where $\mathbf{M} \in \mathbb{R}^{n\times n}$, $\mathbf{C} \in \mathbb{R}^{n\times n}$, $\mathbf{K} \in \mathbb{R}^{n\times n}$, and $\mathbf{D} \in \mathbb{R}^{n\times m}$. As mentioned earlier, $\mathbf{p} \in \mathbb{R}^r$ is a vector of uncertain system parameters, which are functions of the random variable $\boldsymbol{\xi}$ with known probability distribution function $f(\boldsymbol{\xi})$, i.e., $\mathbf{p}=\mathbf{p}(\boldsymbol{\xi})$. It is assumed that the uncertain state vector $\mathbf{x}(t,\mathbf{p})$ and system parameters $M_{ij}$, $C_{ij}$, and $K_{ij}$ can be written as a linear combination of basis functions $\phi_i(\boldsymbol{\xi})$, which span the stochastic space of random variable $\boldsymbol{\xi}$.

$$x_i(t,\mathbf{p}) = \sum_{l=0}^{N} x_{i_l}(t)\phi_l(\boldsymbol{\xi}) = \mathbf{x}_i^T(t)\boldsymbol{\Phi}(\boldsymbol{\xi}) \tag{10}$$

$$M_{ij}(\mathbf{p}) = \sum_{l=0}^{N} m_{ij_l}\phi_l(\boldsymbol{\xi}) = \mathbf{m}_{ij}^T\boldsymbol{\Phi}(\boldsymbol{\xi}) \tag{11}$$

$$C_{ij}(\mathbf{p}) = \sum_{l=0}^{N} c_{ij_l}\phi_l(\boldsymbol{\xi}) = \mathbf{c}_{ij}^T\boldsymbol{\Phi}(\boldsymbol{\xi}) \tag{12}$$

$$K_{ij}(\mathbf{p}) = \sum_{l=0}^{N} k_{ij_l}\phi_l(\boldsymbol{\xi}) = \mathbf{k}_{ij}^T\boldsymbol{\Phi}(\boldsymbol{\xi}) \tag{13}$$

$$D_{ij}(\mathbf{p}) = \sum_{l=0}^{N} d_{ij_l}\phi_l(\boldsymbol{\xi}) = \mathbf{d}_{ij}^T\boldsymbol{\Phi}(\boldsymbol{\xi}) \tag{14}$$

where $\boldsymbol{\Phi}(.) \in \mathbb{R}^N$ is a vector of polynomial basis functions orthogonal to the pdf $f(\boldsymbol{\xi})$, which can be constructed using the *Gram–Schmidt orthogonalization process* [15]. The coefficients $m_{ij_l}$, $c_{ij_l}$, $k_{ij_l}$, and $d_{ij_l}$ are obtained by making use of the following *normal equations* [15]:

$$m_{ij_l} = \frac{\langle M_{ij}(\mathbf{p}(\boldsymbol{\xi})), \phi_l(\boldsymbol{\xi})\rangle}{\langle \phi_l(\boldsymbol{\xi}), \phi_l(\boldsymbol{\xi})\rangle} \tag{15}$$

$$c_{ij_l} = \frac{\langle C_{ij}(\mathbf{p}(\boldsymbol{\xi})), \phi_l(\boldsymbol{\xi})\rangle}{\langle \phi_l(\boldsymbol{\xi}), \phi_l(\boldsymbol{\xi})\rangle} \tag{16}$$

$$k_{ij_l} = \frac{\langle K_{ij}(\mathbf{p}(\boldsymbol{\xi})), \phi_l(\boldsymbol{\xi})\rangle}{\langle \phi_l(\boldsymbol{\xi}), \phi_l(\boldsymbol{\xi})\rangle} \tag{17}$$

$$d_{ij_l} = \frac{\langle D_{ij}(\mathbf{p}(\boldsymbol{\xi})), \phi_l(\boldsymbol{\xi})\rangle}{\langle \phi_l(\boldsymbol{\xi}), \phi_l(\boldsymbol{\xi})\rangle} \tag{18}$$

where $\langle u(\boldsymbol{\xi}), v(\boldsymbol{\xi})\rangle = \int_\Omega u(\boldsymbol{\xi})v(\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}$ represents the inner product introduced by pdf $f(\boldsymbol{\xi})$ with support $\Omega$. Now, the substitution of Eqs. (10)–(14) in Eq. (9) leads to

$$e_i(\boldsymbol{\xi}) = \sum_{j=1}^{n}\left(\sum_{l=0}^{N} m_{ij_l}\phi_l(\boldsymbol{\xi})\right)\left(\sum_{l=0}^{N} \ddot{x}_{j_l}(t)\phi_l(\boldsymbol{\xi})\right) + \sum_{j=1}^{n}\left(\sum_{l=0}^{N} c_{ij_l}\phi_l(\boldsymbol{\xi})\right) \times \left(\sum_{l=0}^{N} \dot{x}_{j_l}(t)\phi_l(\boldsymbol{\xi})\right) + \sum_{j=1}^{n}\left(\sum_{l=0}^{N} k_{ij_l}\phi_l(\boldsymbol{\xi})\right)\left(\sum_{l=0}^{N} x_{j_l}(t)\phi_l(\boldsymbol{\xi})\right) - \sum_{j=1}^{m}\left(\sum_{l=0}^{N} d_{ij_l}\phi_l(\boldsymbol{\xi})\right)u_j, \quad i=1,2,\cdots,n \tag{19}$$

where $e_i(\boldsymbol{\xi})$ is the error in satisfying the system dynamics using Eqs. (10)–(14). Now, $n(N+1)$ time-varying unknown coefficients $x_{i_k}(t)$ can be obtained by using the Galerkin discretization process, i.e., projecting the error of Eq. (19) onto the space of basis functions $\phi_l(\boldsymbol{\xi})$.

$$\langle e_i(\boldsymbol{\xi}), \phi_l(\boldsymbol{\xi})\rangle = 0, \quad i=0,1,2,\ldots,n, \quad l=1,2,\ldots,N \tag{20}$$

where $e_i(\boldsymbol{\xi})$ is the approximation error resulting from representing the system states by a polynomial chaos expansion. This leads to the following set of $n(N+1)$ *deterministic differential equations*:

$$\mathcal{M}\ddot{\mathbf{x}}_p(t) + \mathcal{C}\dot{\mathbf{x}}_p(t) + \mathcal{K}\mathbf{x}_p(t) = \mathcal{D}\mathbf{u}(t) \tag{21}$$

where $\mathbf{x}_p(t)=\{\mathbf{x}_1^T(t),\mathbf{x}_2^T(t),\ldots,\mathbf{x}_n^T(t)\}^T$ is a vector of $n(N+1)$ unknown coefficients and $\mathcal{M}\in\mathbb{R}^{n(N+1)\times n(N+1)}$, $\mathcal{C}\in\mathbb{R}^{n(N+1)\times n(N+1)}$, $\mathcal{K}\in\mathbb{R}^{n(N+1)\times n(N+1)}$, and $\mathbf{D}\in\mathbb{R}^{n(N+1)\times m}$.

Let $P$ and $T_k$, for $k=0,1,2,\ldots,N$, denote the inner product matrices of the orthogonal polynomials defined as follows:

$$P_{ij} = \langle \phi_i(\boldsymbol{\xi}), \phi_j(\boldsymbol{\xi})\rangle, \quad i,j=0,1,2,\ldots,N \tag{22}$$

$$T_{k_{ij}} = \langle \phi_i(\boldsymbol{\xi}), \phi_j(\boldsymbol{\xi}), \phi_k(\boldsymbol{\xi})\rangle, \quad i,j=0,1,2,\ldots,N \tag{23}$$

Then $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{K}$ can be written as $n\times n$ matrix of block matrices, each block being a $(N+1)\times(N+1)$ matrix. The matrix $\mathcal{M}$ consists of blocks $\mathcal{M}_{ij}\in\mathbb{R}^{(N+1)\times(N+1)}$.

$$\mathcal{M}_{ij} = M_{ij}P, \quad i,j=1,2,\ldots,n \tag{24}$$

If the mass matrix is not uncertain, else, it is given by

$$\mathcal{M}_{ij}(k,:) = \mathbf{m}_{ij}^T T_k, \quad i,j=1,2,\ldots,n \tag{25}$$

Similarly, for the matrices $\mathcal{C}$ and $\mathcal{K}$, the $k$th row of each of their block matrices $\mathcal{C}_{ij},\mathcal{K}_{ij}\in\mathbb{R}^{(N+1)\times(N+1)}$ is given by

$$\mathcal{C}_{ij}(k,:) = \mathbf{c}_{ij}^T T_k, \quad i,j=1,2,\ldots,n \tag{26}$$

$$\mathcal{K}_{ij}(k,:) = \mathbf{k}_{ij}^T T_k, \quad i,j=1,2,\ldots,n \tag{27}$$

The matrix $\mathcal{D}$ consists of blocks $\mathcal{D}_{ij}\in\mathbb{R}^{(N+1)\times 1}$.

$$\mathcal{D}_{ij} = P\mathbf{d}_{ij} \quad i=1,2,\ldots,n, \quad j=1,2,\ldots,m \tag{28}$$

Equation (10) along with Eq. (21) define the uncertain state vector $\mathbf{x}(t,\boldsymbol{\xi})$ as a function of random variable $\boldsymbol{\xi}$ and can be used to compute any order moment or cumulant of a function of uncertain state variable. For example, the first moment of residual energy can be computed as:

$$\mathbf{E}[V](T_Z) = \int_\Omega V(T_Z,\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi} = \frac{1}{2}\dot{\mathbf{x}}_p^T(T_Z)\mathcal{M}\dot{\mathbf{x}}_p(T_Z) + \frac{1}{2}\mathbf{x}_p^T(T_Z)\mathcal{K}(\mathbf{p})\mathbf{x}_p(T_Z) - \frac{1}{2}\mathbf{E}[\mathbf{x}_f^T\mathbf{x}_f - 2\mathbf{x}^T\mathbf{x}_f] \tag{29}$$

The problem of the design of a robust input shaper as described in Sec. 2 is one, which minimizes the $\ell_\infty$ norm of residual energy over the domain of uncertain parameters. In a probabilistic framework, this corresponds to concurrently minimizing the mean and the support of the probability distribution of the residual energy. This can be achieved by posing a multi-objective optimization problem. The cost function is a weighted sum of the absolute values of the central moments. This cost function can be represented as

$$\min_{A_i,T_i}\left(\alpha_1\mathbf{E}[V(T_Z)] + \sum_{i=2}^{P}\alpha_i\left|\mathbf{E}[(V(T_Z)-\mathbf{E}[V(T_Z)])^i]\right|\right) \tag{30}$$

where $\alpha_i > 0$ is a weighting parameter. Notice that $P=2$ corresponds to minimization of mean and variance of the residual energy. One can add higher moments such as skew and kurtosis and minimizing the skew and kurtosis is consistent with the goal of minimizing the worst cost over the domain of uncertainty. In the next section, we illustrate the proposed procedure by considering two examples.

## 4 Example (Single Spring-Mass System)

To illustrate the proposed technique of using polynomial chaos for the determination of a robust input shaper, we consider the second order system

$$\ddot{x} + kx = ku \tag{31}$$

where $k$ is an uncertain parameter of the system, which is known to lie in the interval $[a\ b]$. We assume it to be a function of random variable $\xi$ with known probability density function $f(\xi)$. Thus, the uncertain parameter $k$ can be represented as

$$k(\xi) = \sum_{i=0}^{N} k_i\phi_i(\xi) \tag{32}$$

Furthermore, if $\xi\in[-1,1]$, only two terms are necessary to represent $k(\xi)$.

$$k(\xi) = k_0 + k_1\xi, \quad k_0 = \frac{a+b}{2}, \quad k_1 = \frac{b-a}{2} \tag{33}$$

This does not preclude normal distributions since $k_0$ and $k_1$ can represent the mean and standard deviation of $k(\xi)$ and $\xi\in(-\infty,\infty)$. Now, the displacement $x$ is represented as

$$x = \sum_{i=0}^{N} x_i(t)\phi_i(\xi) \tag{34}$$

where $\phi_i(\xi)$ represents the orthogonal polynomial set with respect to pdf $f(\xi)$, i.e.,

$$\langle \phi_i(\xi), \phi_j(\xi)\rangle = \int_\Omega \phi_i(\xi)\phi_j(\xi)f(\xi)d\xi = c_i^2\delta_{ij} \tag{35}$$

For example, the Legendre and the Hermite polynomials constitute the orthogonal polynomial sets for uniform and normal distributions, respectively. In general, these polynomials can be constructed by making use of Gram–Schmidt orthogonalization process. Now, substituting for $x$ and $k$ from Eqs. (34) and (32) in Eq. (31) leads to

$$\sum_{i=0}^{N}\phi_i(\xi)\ddot{x}_i + (k_0\phi_0(\xi) + k_1\phi_1(\xi))\sum_{i=0}^{N}\phi_i(\xi)x_i = (k_0\phi_0(\xi) + k_1\phi_1(\xi))u \tag{36}$$

Using the Galerkin projection method, the dynamics of $x_i$ can be determined. Making use of the fact that system equation error due to polynomial chaos approximation (Eq. (36)) should be orthogonal to basis function set $\phi_j(\xi)$, we arrive at the equation

$$\mathcal{M}\begin{Bmatrix}\ddot{x}_0\\ \ddot{x}_1\\ \vdots\\ \ddot{x}_N\end{Bmatrix} + \mathcal{K}\begin{Bmatrix}x_0\\ x_1\\ \vdots\\ x_N\end{Bmatrix} = \mathcal{D}u \tag{37}$$

where the elements of the $\mathcal{M}$ matrix are

**Table 1 Uniform distribution**

| Index | Basis functions ($P_i$) | $a_i$ | $c_i^2$ |
|---|---|---|---|
| 0 | 1 | 1 | 2 |
| 1 | $\xi$ | 1 | $\frac{2}{3}$ |
| 2 | $\frac{1}{2}(3\xi^2-1)$ | $\frac{3}{2}$ | $\frac{2}{5}$ |
| 3 | $\frac{1}{2}(5\xi^3-3\xi)$ | $\frac{5}{2}$ | $\frac{2}{7}$ |

$$\mathcal{M}_{ij} = \langle \phi_i(\xi), \phi_j(\xi) \rangle = \int_\Omega \phi_i(\xi)\phi_j(\xi) f(\xi) d\xi = c_i^2 \delta_{ij} \quad \text{where} \quad i,j = 0,1,2,\ldots,N \tag{38}$$

and the elements of the $\mathcal{K}$ matrix are given by

$$\mathcal{K}_{ij} = k_0 \langle \phi_i(\xi), \phi_j(\xi) \rangle + k_1 \langle \xi \phi_i(\xi), \phi_j(\xi) \rangle \tag{39}$$

Now, making use of the fact that every orthogonal polynomial set satisfies a three-term recurrence relation [16].

$$\xi \phi_n(\xi) = \frac{a_n}{a_{n+1}} \phi_{n+1}(\xi) + \frac{c_n^2}{c_{n-1}^2} \frac{a_{n-1}}{a_n} \phi_{n-1}(\xi) \tag{40}$$

where $a_n$ and $a_{n-1}$ are the leading coefficients of $\phi_n(\xi)$ and $\phi_{n-1}(\xi)$, respectively. Now, making use of this recurrence relationship, the elements of the $\mathcal{K}$ matrix are given by

$$\mathcal{K}_{ii} = k_0 \langle \phi_i(\xi), \phi_j(\xi) \rangle = k_0 c_i^2 \tag{41}$$

$$\mathcal{K}_{i,i+1} = k_1 \langle \phi_{i+1}(\xi), \phi_j(\xi) \rangle = k_1 c_{i+1}^2 \frac{a_i}{a_{i+1}} \tag{42}$$

$$\mathcal{K}_{i,i-1} = k_1 \frac{c_i^2}{c_{i-1}^2} \langle \phi_{i-1}(\xi), \phi_j(\xi) \rangle = k_1 c_i^2 \frac{a_{i-1}}{a_i} \tag{43}$$

and

$$\mathcal{D} = \begin{bmatrix} c_0^2 k_0 \\ c_1^2 k_1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{44}$$

Equation (37) can be easily solved for a parameterized $u$ (Eq. (5)). The residual energy at the final time $T_Z$ can be represented as

$$V(T_Z,\xi) = \frac{1}{2}\left(\sum_{i=0}^N \dot{x}_i \phi_i(\xi)\right)^T \left(\sum_{i=0}^N \dot{x}_i \phi_i(\xi)\right) + \frac{1}{2}\left(\sum_{i=0}^N x_i \phi_i(\xi) - x_f\right)^T k(\xi) \left(\sum_{i=0}^N x_i \phi_i(\xi) - x_f\right) \tag{45}$$

The mean of the residual energy can be easily calculated using the equation

$$\mathbf{E}[V(T_Z,\xi)] = \mu = \int_\Omega V(T_Z,\xi) f(\xi) d\xi \tag{46}$$

and the higher central moments by the equation

$$\mathbf{E}[(V(T_Z,\xi)-\mu)^n] = \int_\Omega (V(T_Z,\xi)-\mu)^n f(\xi) d\xi \tag{47}$$

**4.1 Uniform Distribution.** To illustrate the proposed technique of using polynomial chaos for the determination of a robust input shaper, consider the spring-mass system (Eq. (31)), where $k$ is a uniformly distributed random variable, which lies in the range

$$\underbrace{0.7}_{a} \le k \le \underbrace{1.3}_{b} \tag{48}$$

The random variable $k$ can be represented by Legendre polynomials ($P_i$) as

$$k(\xi) = k_0 P_0(\xi) + k_1 P_1(\xi) \tag{49}$$

where $\xi$ is a uniformly distributed random variable, which lies in the range $[-1\ ,1]$ and $k_0=(a+b)/2=1$ and $k_1=(b-a)/2=0.3$. Table 1 lists the basis functions and the coefficients, which are necessary to construct the polynomial chaos coefficient dynamics, which can be represented as

$$\begin{Bmatrix} \ddot{x}_0 \\ \ddot{x}_1 \\ \vdots \\ \ddot{x}_{i-1} \\ \ddot{x}_i \\ \ddot{x}_{i+1} \\ \vdots \\ \ddot{x}_N \end{Bmatrix} + \begin{bmatrix} k_0 & \frac{1}{3}k_1 & 0 & & & & & & \\ k_1 & k_0 & \frac{2}{5}k_1 & & & & & & \\ 0 & \ddots & \ddots & \ddots & & & & & \\ & & & k_0 & \frac{i}{2i+1}k_1 & 0 & & & \\ & & & \frac{i}{2i-1}k_1 & k_0 & \frac{(i+1)}{2i+3}k_1 & & & \\ & & & 0 & \frac{i+1}{2i+1}k_1 & k_0 & & & \\ & & & & & & \ddots & \ddots & \frac{N}{2N+1}k_1 \\ & & & & 0 & 0 & & \frac{n}{2N-1}k_1 & k_0 \end{bmatrix} \begin{Bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{i-1} \\ x_i \\ x_{i+1} \\ \vdots \\ x_N \end{Bmatrix} = \begin{bmatrix} k_0 \\ k_1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \tag{50}$$

**Table 2 Optimal input shaper (two delays)**

| Cost | $A_0$ | $A_1$ | $A_2$ | $T_1$ | $T_2$ |
|---|---|---|---|---|---|
| $\mathbf{E}[V(T_2)]=\mu$ | 0.2545 | 0.4909 | 0.2545 | 3.1416 | 6.2831 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}$ | 0.2554 | 0.4892 | 0.2554 | 3.1416 | 6.2830 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}+\lvert\sqrt[3]{\mathbf{E}[(V(T_2)-\mu)^3]}\rvert$ | 0.2557 | 0.4886 | 0.2557 | 3.1415 | 6.2836 |

Equation (50) can be easily solved for a parameterized $u$ (Eq. (5)). The residual energy at the final time $T_Z$ can be represented as

$$V(T_Z,\xi)=\frac{1}{2}\left(\sum_{i=0}^{N}\dot{x}_iP_i(\xi)\right)^T\left(\sum_{i=0}^{N}\dot{x}_iP_i(\xi)\right)+\frac{1}{2}\left(\sum_{i=0}^{N}x_iP_i(\xi)-x_f\right)^T k(\xi)\left(\sum_{i=0}^{N}x_iP_i(\xi)-x_f\right) \quad (51)$$

The mean of the residual energy can be easily calculated using the equation

$$\mathbf{E}[V(T_Z,\xi)]=\mu=\int_{\Omega}V(T_Z,\xi)\frac{1}{2}d\xi \quad (52)$$

and the higher central moments by the equation

$$\mathbf{E}[(V(T_Z,\xi)-\mu)^n]=\int_{\Omega}(V(T_Z,\xi)-\mu)^n\frac{1}{2}d\xi \quad (53)$$

Parameterize a two time-delay filter as

$$u=A_0+A_1\mathcal{H}(t-T_1)+A_2\mathcal{H}(t-T_2) \quad (54)$$

with the constraint that $A_0+A_1+A_2=1$ to mandate the final value of the output of the time-delay filter subject to a step input will be the same as the magnitude of the input. The polynomial chaos expansion of eighth order is used to represent the residual energy random variable $V(T_2)$. A constrained minimization problem is solved to minimize a series of cost functions, which are listed in Table 2. Figure 1 illustrates the performance of the PC based design to the standard minimax controller design, which endeavors to minimize the maximum magnitude of the residual energy over the domain of spring stiffness uncertainty. The distribution of the uncertain spring stiffness is assumed to be uniform. It is clear from Fig. 1 that as higher moments are included in the design process, the resulting solutions tend toward the minimax solution. This is further evidenced in Fig. 2, where a three time-delay input shaper design based on polynomial chaos and the minimax approach are compared (Table 3 lists the optimal solutions).

To illustrate the ability of the polynomial chaos expansion to capture the distribution of the residual energy, the random coefficient of stiffness is sampled with 10,000 Monte Carlo samples and the distribution of the residual energy is calculated. The mean and variance of the Monte Carlo based sampling is compared with those of the polynomial chaos based expansion. These results are illustrated in Fig. 3, where the Monte Carlo based result is illustrated by the dashed line. The solid line is the estimate of the mean and variance as a function of the number of terms in the polynomial chaos expansion. It is clear that with a fourth order polynomial, the mean and variance have converged to the true values.

**4.2 Gaussian Distribution.** One can determine a robust input shaper for normally distributed coefficient of stiffness similarly. Consider a random stiffness coefficient given by the equation

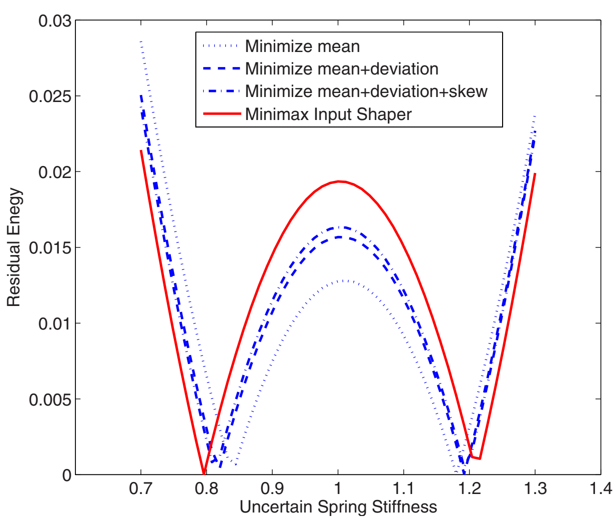



**Fig. 1 PC uniform distribution (two delays filter)**

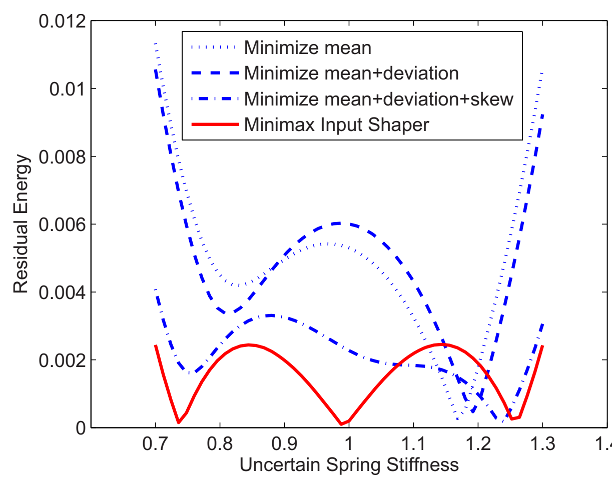



**Fig. 2 PC uniform distribution (three delays filter)**

**Table 3 Optimal input shaper (three delays)**

| Cost | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|---|---|---|---|
| $\mathbf{E}[V(T_2)]=\mu$ | 0.1810 | 0.4167 | 0.3223 | 0.0799 | 3.1365 | 6.2869 | 9.4237 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}$ | 0.1810 | 0.4160 | 0.3231 | 0.0800 | 3.1380 | 6.2858 | 9.4240 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}+\lvert\sqrt[3]{\mathbf{E}[(V(T_2)-\mu)^3]}\rvert$ | 0.1357 | 0.3747 | 0.3658 | 0.1238 | 3.1500 | 6.2926 | 9.4190 |

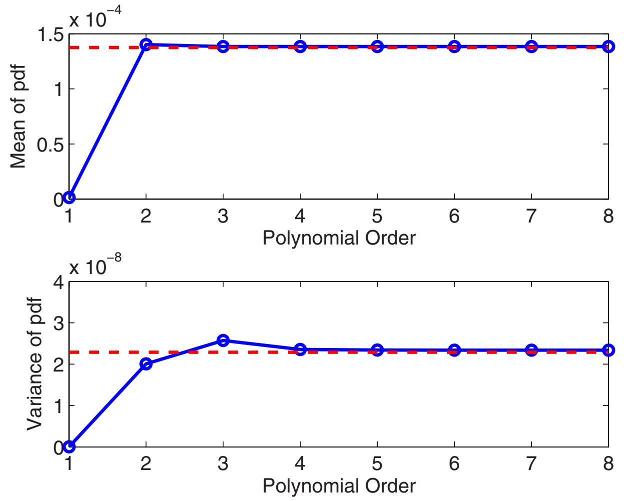



**Fig. 3 Polynomial chaos based estimate of mean and variance**

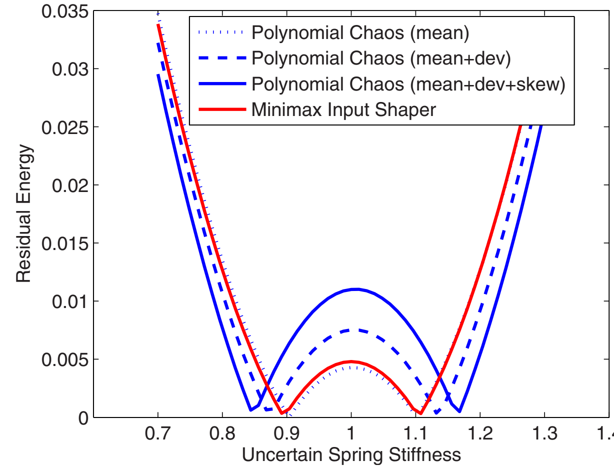



**Fig. 4 PC gaussian distribution (two delays filter)**

$$k = \mathcal{N}(1, 0.1^2) \tag{55}$$

which corresponds to a Gaussian distribution with a mean of unity and a standard deviation of $\sigma=0.1$. The variable $k$ can also be represented using Hermite polynomials as

$$k = k_0 H_0(\xi) + k_1 H_1(\xi) \tag{56}$$

where $k_0=1$ and $k_1=0.05$. Table 4 lists the Hermite polynomials, which are the orthogonal basis functions for a normally distributed random variable.

Assuming a Gaussian distributed random spring stiffness with a mean of unity and a deviation of 0.1, the residual energy represented by the polynomial chaos is determined and different combinations of the moments are minimized. The minimax problem is also solved to provide a benchmark to compare the solution of the polynomial chaos expansion.

Figures 4 and 5 illustrate the results of the optimization problem, which minimizes the mean, mean+deviation, and mean+deviation+skew and compares it to the weighted minimax solution, which is considered the desired solution. It is clear for both the two time-delay filter in Fig. 4 and the three time-delay filter in Fig. 5 that minimizing the mean results in a solution that is closest to the weighted minimax solution. Tables 5 and 6 list the optimal time-delay filter parameters for a two time-delay and three time-delay filter, respectively.

**4.3 Compact Support Polynomial Distribution.** Uncertain parameters such as spring stiffness, mass, damping factors, etc., do not have infinite support since they are constrained to be positive. Thus, the Gaussian distribution is not an appropriate distribution to represent the uncertainty in these parameters. Gaussian like distributions with compact support were developed by Singla and Junkins [16] using special polynomial functions to blend diverse local models into a statistically unbiased global representation. These specially designed functions smoothly go to zero at the end points of their compact support and have a maximum at the center of the support. These functions are

**Table 4 Normal distribution**

| Index | Basis functions ($H_i$) | $a_i$ | $c_i^2$ |
|---|---|---|---|
| 0 | $1$ | 1 | $\sqrt{2\pi}$ |
| 1 | $\xi$ | 1 | $4\sqrt{2\pi}$ |
| 2 | $\xi^2-1$ | 1 | $9\sqrt{2\pi}$ |
| 3 | $\xi^3-3\xi$ | 1 | $16\sqrt{2\pi}$ |

$$w(t) = 1 - K\sum_{n=0}^{m} A_n \xi^{2m-n+1} \tag{57}$$

where $K$ and $A_n$ are given by the following expressions:

$$K = \frac{(2m+1)!(-1)^m}{(m!)^2}, \quad A_n = \frac{(-1)^{nm} C_n}{2m-n+1}, \quad {}^m C_n = \frac{m!}{n!(m-n)!} \tag{58}$$

and will be used in the design of robust input shapers in this work.

Figure 6 illustrates five of these functions with the order of polynomial ranging from 0, which corresponds to the triangular shape to ten which tends, to a *top-hat* function. This suite of functions are positive over the domain $\xi \in [-1,1]$ and have an enclosed area of one permitting them to be used as probability density functions.

For illustrative purposes, assume that the pdf of the stiffness is

$$f(\xi) = 1 - \xi^2(3 - 2|\xi|) \tag{59}$$

The Gram–Schmidt process can be used to determine the orthogonal basis functions $\phi_i$. Assume that the basis functions, which we consider to construct the orthogonal polynomial bases are

$$\psi_i(\xi) = \xi^i \quad i = 0,1,2,\ldots \tag{60}$$

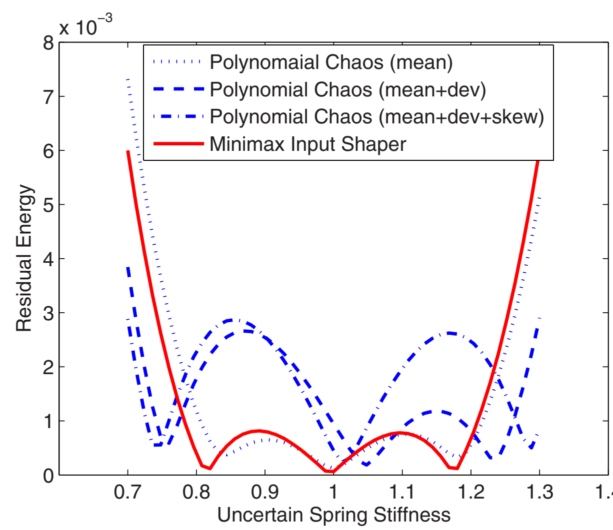



**Fig. 5 PC Gaussian distribution (three delays filter)**

**Table 5 Optimal input shaper (two delays)**

| Cost | $A_0$ | $A_1$ | $A_2$ | $T_1$ | $T_2$ |
|---|---|---|---|---|---|
| $\mathbf{E}[V(T_2)]=\mu$ | 0.2515 | 0.4970 | 0.2515 | 3.1416 | 6.2831 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}$ | 0.2527 | 0.4947 | 0.2526 | 3.1416 | 6.2830 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}+\lvert\sqrt[3]{\mathbf{E}[(V(T_2)-\mu)^3]}\rvert$ | 0.2540 | 0.4923 | 0.2537 | 3.1417 | 6.2829 |

Assume the first basis $\phi_0=\psi_0=1$. The remaining basis $\phi_i(\xi)$ can be constructed using the equation

$$\phi_i(\xi)=\psi_i(\xi)-\sum_{j=1}^{i-1}\frac{\langle\psi_i(\xi),\phi_j(\xi)\rangle}{\langle\phi_j(\xi),\phi_j(\xi)\rangle}\phi_j(\xi),\quad i=1,2,3,\ldots \tag{61}$$

Table 7 lists the orthonormal basis for the pdf given by Eq. (59). Assuming a spring stiffness distribution given by the equation

$$k(\xi)=k_0G_0(\xi)+k_1G_1(\xi)=1+0.3\xi \tag{62}$$

and solving for the input shaper, which minimizes the mean of the distribution of the residual energy and its higher moments, results in the shaper parameters presented in Tables 8 and 9.

Figures 7 and 8 illustrate the results of the optimization problem, which minimizes the mean, mean+deviation, and mean+deviation+skew and compares it to the weighted minimax solution, which is considered the desired solution.

It is clear for both the two time-delay filter in Fig. 7 and the three time-delay filter in Fig. 8 that minimizing the mean results in a solution that is closest to the weighted minimax solution.

To illustrate that the approximation of the pdf of the residual energy using the polynomial chaos approach closely matches the Monte Carlo results and to illustrate the benefit of using higher central moments in the optimization algorithm to drive the solution of the optimizer toward the minimax solution, four sets of graphs are presented in Fig. 9. The first column presents histograms of residual energy generated by a Monte Carlo simulation and the second column is the polynomial chaos' counterpart. Ten thousand samples are used to generate the histograms. The first row corresponds to the distribution when the mean of residual energy is minimized. The second, third, and fourth rows correspond to minimization of the first 3, 5, and 50 central moments, respectively. It is evident from these figures that minimizing the mean of the residual energy results in a delta distribution with a long tail, i.e., large support. Moreover, with an increase in the number of central moments included in the optimization, the shape of the pdf tends to reduce the support and uniformly distribute the residual energy, resulting in a solution, which is similar to the minimax problem ($\ell_\infty$ norm minimization).

If one is concerned with minimizing a pdf weighted residual energy function rather than the worst case problem, one needs to only minimize the mean of the residual energy using the polynomial chaos approach. If one is interested in the worst case design, then one needs to include multiple central moments in the cost.

## 5 Example (Three Mass-Spring System)

To illustrate the use of polynomial chaos expansion for solving robust input shaper design problems for systems with multiple uncertain parameters, we consider a three mass-spring system with two random variables. The three mass-spring system illustrated in Fig. 10 can represent a simplified model of a double-pendulum crane. The variation in the cable length is represented as an uncertainty in the coefficient of spring stiffness $k_1$. The variation in load carried by the crane is represented by assuming that the mass $m_3$ is uncertain. The system model is

$$\begin{bmatrix} m_1 & 0 & 0\\ 0 & m_2 & 0\\ 0 & 0 & m_3\end{bmatrix}\begin{Bmatrix}\ddot{y}_1\\ \ddot{y}_2\\ \ddot{y}_3\end{Bmatrix}+\begin{bmatrix} k_1 & -k_1 & 0\\ -k_1 & k_1+k_2 & -k_2\\ 0 & -k_2 & k_2\end{bmatrix}\begin{Bmatrix} y_1\\ y_2\\ y_3\end{Bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}u \tag{63}$$

Assume that the constant system parameters are

**Table 6 Optimal input shaper (three delays)**

| Cost | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|---|---|---|---|
| $\mathbf{E}[V(T_2)]=\mu$ | 0.1276 | 0.3729 | 0.3725 | 0.1271 | 3.1433 | 6.2813 | 9.4145 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}$ | 0.1229 | 0.3694 | 0.3768 | 0.1309 | 3.1409 | 6.2792 | 9.4153 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}+\lvert\sqrt[3]{\mathbf{E}[(V(T_2)-\mu)^3]}\rvert$ | 0.1179 | 0.3629 | 0.3817 | 0.1374 | 3.1421 | 6.2780 | 9.4061 |

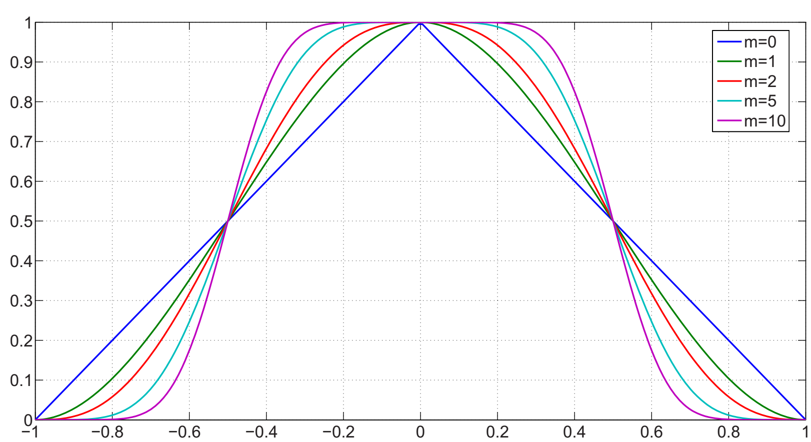



**Fig. 6 Compact support distributions**

**Table 7 Compact support distribution**

| Index | Basis functions $(G_i)$ | $a_i$ | $c_i^2$ |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | $\xi$ | 1 | $\frac{2}{15}$ |
| 2 | $\xi^2-\frac{2}{15}$ | 1 | $\frac{79}{3150}$ |
| 3 | $\xi^3-\frac{9}{28}\xi$ | 1 | $\frac{31}{5880}$ |

$$m_1=1, \quad m_2=1, \quad k_2=1 \tag{64}$$

and the uncertain parameters $k_1$ and $m_3$ are assumed to be represented as random variables

$$k_1 \sim f_1(\xi_1) \quad \text{and} \quad m_3 \sim f_2(\xi_2) \tag{65}$$

with probability density functions $f_1(\xi_1)$ and $f_2(\xi_2)$, respectively. The desire is to design a shaped reference profile, which is robust to uncertainties in $k_1$ and $m_3$ and will be achieved by representing the system response to a parameterized reference input using polynomial chaos.

Since the system includes a rigid body mode, a PD controller

$$u=-k_p(y_1-y_r)-k_d\dot{y}_1 \tag{66}$$

is used to stabilize the system. The resulting closed-loop system is

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{\mathbf{M}} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + \underbrace{\begin{bmatrix} k_d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{C}} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{Bmatrix}$$
$$+ \underbrace{\begin{bmatrix} k_1+k_p & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} k_p \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{D}} y_r \tag{67}$$

The reference input $y_r$ is shaped by filtering a unit step input through a time-delay filter parameterized using $A_i$ and $T_i$ as

$$Y_R(s)=\frac{1}{s}\sum_{i=0}^{Z} A_i \exp(-sT_i) \tag{68}$$

where $T_0$=0. A robust filter, which minimizes the worst performance of the system over the domain of uncertainties in $m_3$ and $k_1$ is posed as an optimization problem when the expected mean and higher central moments of the distribution of the residual energy are minimized. To ensure that the final value of the shaped profile is the same as the reference input, we require

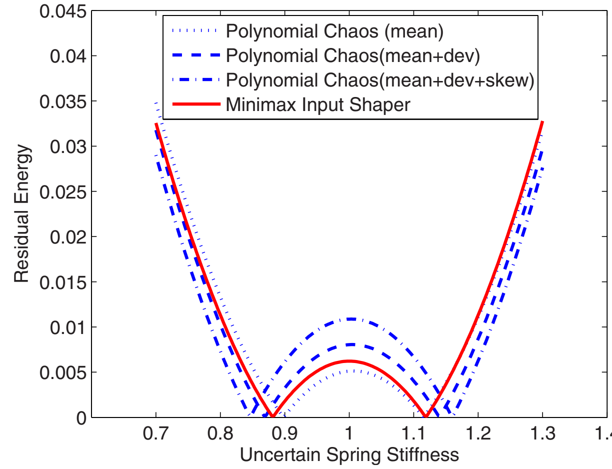



**Fig. 7 PC compact polynomial distribution (two delays filter)**

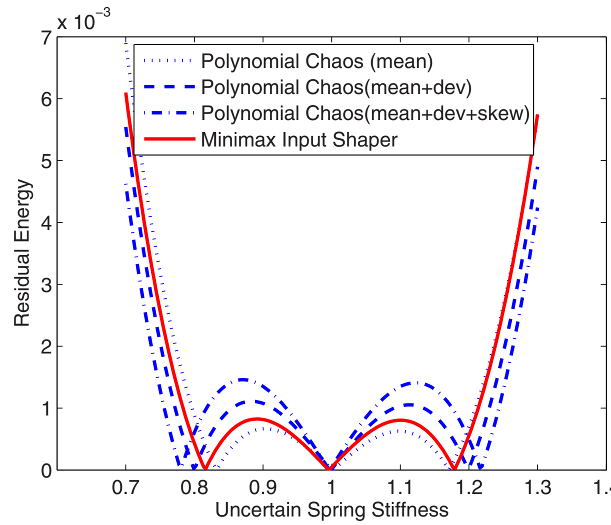



**Fig. 8 PC compact polynomial distribution (three delays filter)**

$$\sum_{i=0}^{Z} A_i = 1 \tag{69}$$

The uncertain model parameters are represented as a vector $\mathbf{p}$ of the uncorrelated random variables $k_1$ and $m_3$ of known pdfs $f(\boldsymbol{\xi})$, where $\boldsymbol{\xi}$ is a two-dimensional vector. The random variable $\mathbf{p}$ can be expanded using polynomial chaos, permitting the states and the elements of the mass, damping, stiffness, and control influence

**Table 8 Optimal input shaper (two delays)**

| Cost | $A_0$ | $A_1$ | $A_2$ | $T_1$ | $T_2$ |
|---|---|---|---|---|---|
| $\mathbf{E}[V(T_2)]=\mu$ | 0.2518 | 0.4964 | 0.2518 | 3.1358 | 6.27170 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}$ | 0.2528 | 0.4943 | 0.2528 | 3.1418 | 6.2835 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}+\lvert\sqrt[3]{\mathbf{E}[(V(T_2)-\mu)^3]}\rvert$ | 0.2538 | 0.4923 | 0.2538 | 3.1464 | 6.2929 |

**Table 9 Optimal input shaper (three delays)**

| Cost | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|---|---|---|---|
| $\mathbf{E}[V(T_2)]=\mu$ | 0.1273 | 0.3728 | 0.3727 | 0.1273 | 3.1463 | 6.2928 | 9.4391 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}$ | 0.1282 | 0.3718 | 0.3718 | 0.1282 | 3.1532 | 6.3069 | 9.4602 |
| $\mu+\sqrt{\mathbf{E}[(V(T_2)-\mu)^2]}+\lvert\sqrt[3]{\mathbf{E}[(V(T_2)-\mu)^3]}\rvert$ | 0.1289 | 0.3711 | 0.3711 | 0.1289 | 3.1573 | 6.3153 | 9.4725 |

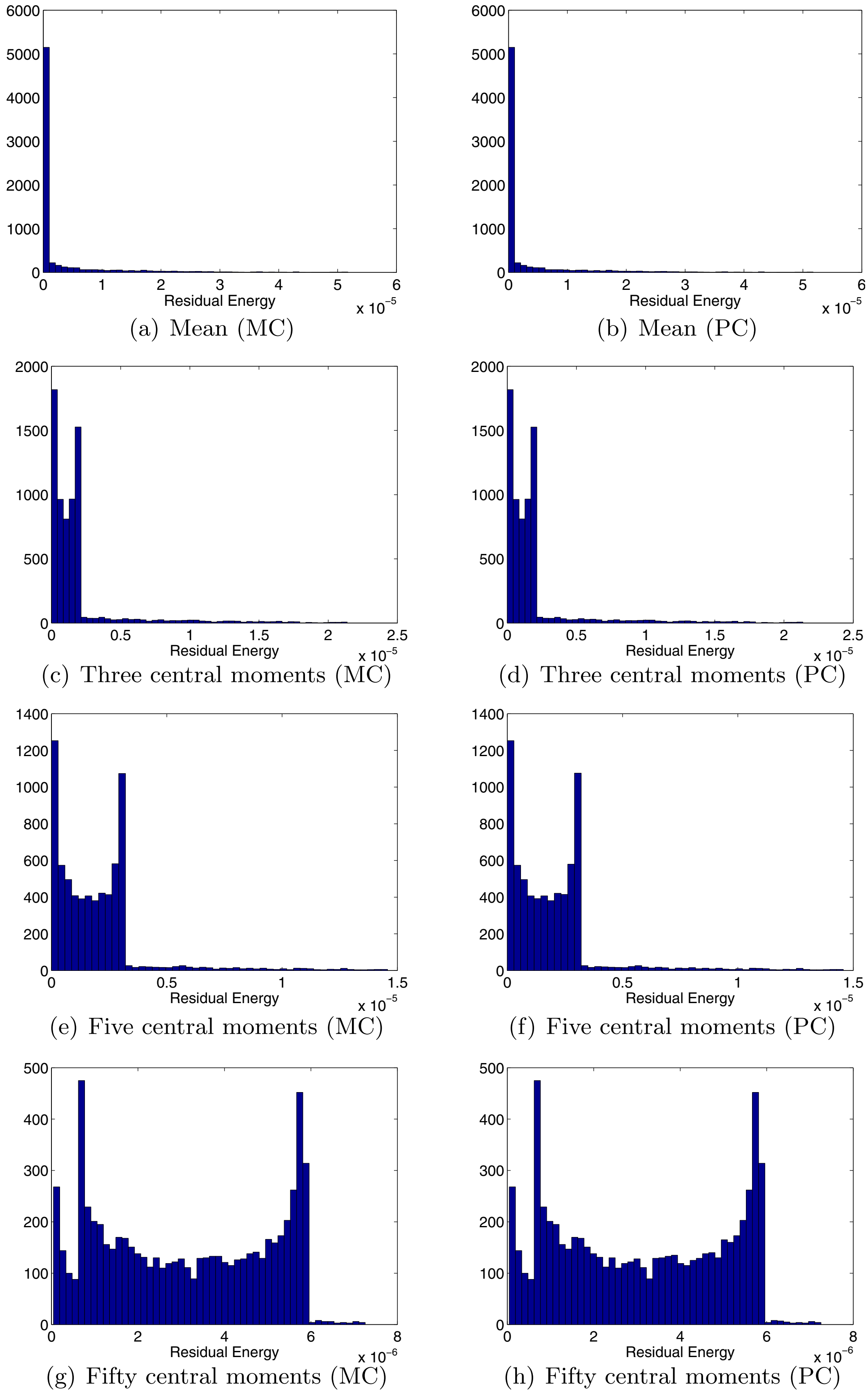



**Fig. 9 MC versus PC**

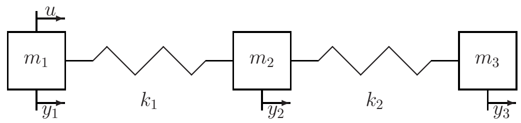



**Fig. 10 Three mass-spring systems**

matrices to be represented using Eqs. (10)–(14). The orthogonal polynomials used in the polynomial chaos expansion are constructed by the tensor product of 1D polynomials in the $\xi_1$ and $\xi_2$ space assuming the joint pdf of the random variable is given by the equation

$$f(\boldsymbol{\xi}) = f_1(\xi_1) f_2(\xi_2) \tag{70}$$

The resulting orthogonal polynomials are given as

$$\phi_{ij}(\xi_1,\xi_2) = \phi_i(\xi_1)\phi_j(\xi_2), \quad \forall\, i = 0,1,\ldots,N, \quad j = 0,1,\ldots,N, \quad \text{where} \quad i+j \le N \tag{71}$$

Figure 11 illustrates the first ten two-dimensional orthogonal basis functions for the probability density function $f(\xi_1,\xi_2) = (1-\xi_1^2(3-2|\xi_1|))(1-\xi_2^2(3-2|\xi_2|))$.

The dynamic model for the coefficients of the polynomials expansion used to represent the system states are derived using the Galerkin projection method described in Sec. 3 exploiting the property that

$$\langle \phi_{ij}(\xi_1,\xi_2), \phi_{lm}(\xi_1,\xi_2)\rangle = \int_{\Omega_1}\int_{\Omega_2} f(\xi_1) f(\xi_2)\phi_{ij}(\xi_1,\xi_2)\phi_{lm}(\xi_1,\xi_2)d\xi_1 d\xi_2 \tag{72}$$

$$= \int_{\Omega_1} f(\xi_1)\phi_i(\xi_1)\phi_l(\xi_1)d\xi_1 \int_{\Omega_2} f(\xi_2)\phi_j(\xi_2)\phi_m(\xi_2)d\xi_2 \tag{73}$$

$$= c_i^2 c_j^2 \delta(i-l)\delta(j-m) \tag{74}$$

The robust input shaping design is illustrated assuming that the uncertain parameters $k_1$ and $m_3$ can be represented either by a Gaussian distribution

$$k_1 \sim \mathcal{N}[1, 0.04] \quad \text{and} \quad m_3 \sim \mathcal{N}[2, 0.16] \tag{75}$$

or a uniform distribution

$$k_1 \sim \mathcal{U}[0.7, 1.3] \quad \text{and} \quad m_3 \sim \mathcal{U}[1.4, 2.6] \tag{76}$$

**5.1 Uniform Distribution.** Representing the uncertain parameters for a uniformly distributed density function as

$$k_1 = k_1^0 P_0(\xi_1) + k_1^1 P_1(\xi_1) \tag{77}$$

and

$$m_3 = m_3^0 P_0(\xi_2) + m_3^1 P_1(\xi_2) \tag{78}$$

where $P_i$ are Legendre polynomials and $\xi_1$ and $\xi_2$ lie in the range $[-1, 1]$, we have $k_1^0 = (a+b)/2 = 1$ and $k_1^1 = (b-a)/2 = 0.3$ and $m_3^0 = 2$ and $m_3^1 = 0.6$.

Assuming $N=2$, the system states can be represented in terms of Legendre polynomials, which are functions of the two random variables as

$$x_i(t,\xi_1,\xi_2) = x_i^0(t) + x_i^1(t)\xi_1 + x_i^2(t)\xi_2 + x_i^3(t)\frac{1}{2}(3\xi_1^2 - 1) + x_i^4(t)\frac{1}{2}(3\xi_2^2 - 1) + x_i^5(t)\xi_1\xi_2 \tag{79}$$

The Galerkin projection leads to the equation

$$\mathcal{M}\ddot{\mathcal{X}} + \mathcal{C}\dot{\mathcal{X}} + \mathcal{K}\mathcal{X} = \mathcal{D}x_r \tag{80}$$

where $\mathcal{X} \in \mathbb{R}^{n(N+1)(N+2)/2}$. $n$ and $N$ refer to the number of states and order of polynomials used in the expansion, respectively. For the design of the input shaper $N$, the order of polynomial in one dimension is selected to be four. The shaped reference input is parameterized using a seven time-delay filter and three cost functions, which are functions of central moments of the residual en-

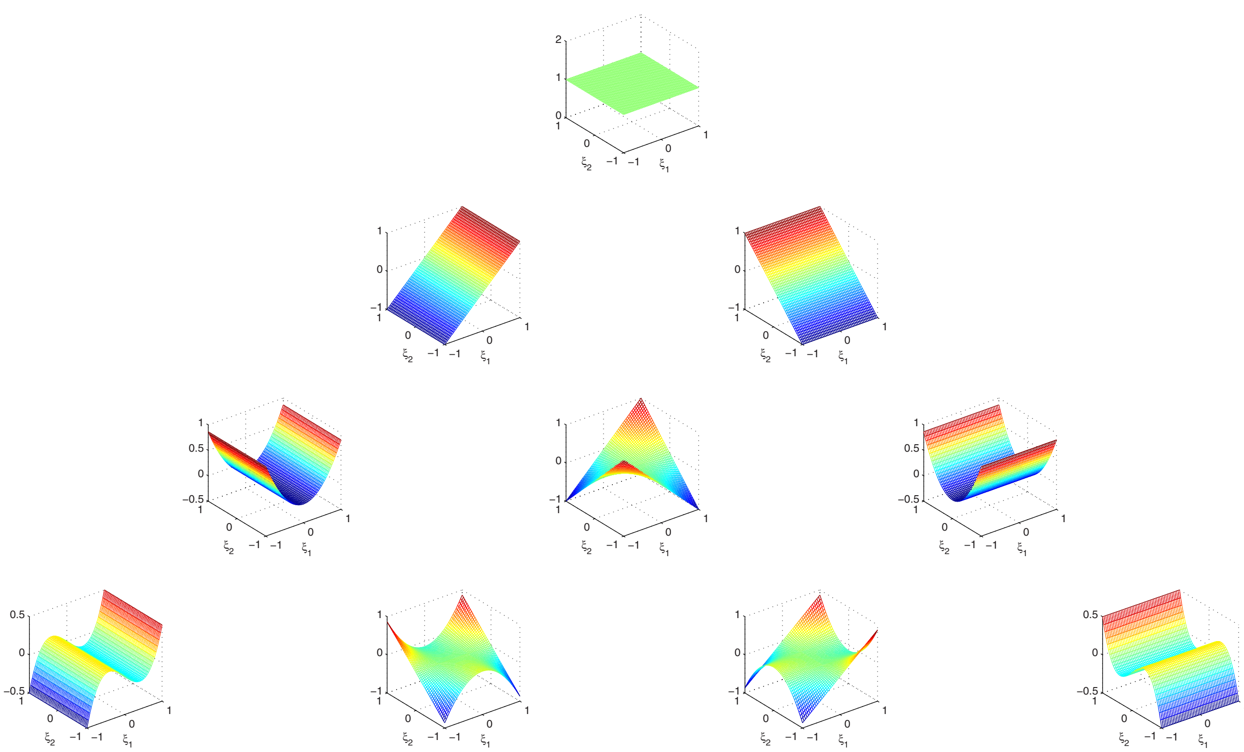
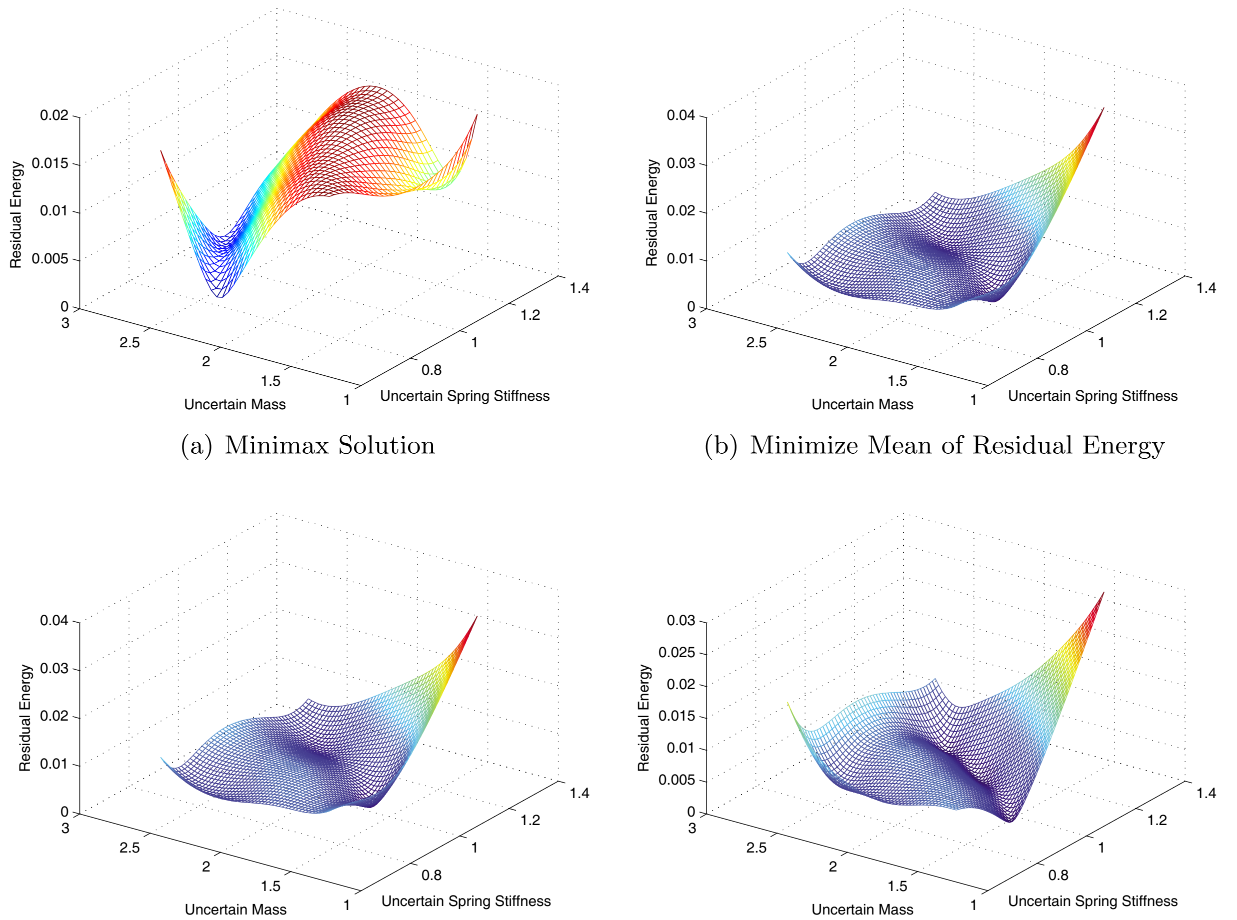


**Fig. 11 Two-dimensional orthogonal basis functions**




(a) Minimax Solution

(b) Minimize Mean of Residual Energy

(c) Minimize Mean+Deviation of Residual Energy

(d) Minimize Mean+Deviation+Skew of Residual Energy

**Fig. 12 Residual energy distribution**

ergy are minimized. A minimax optimization problem is solved, where the largest magnitude of the residual energy over the domain of uncertain parameters $(k_1, m_3)$ is minimized to permit comparison with the performance of the polynomial chaos based robust design. Figure 12 illustrates the variation in the residual energy over the uncertain domain. Figure 12(*a*) corresponds to the minimax solution and is used as the benchmark to compare the other three graphs. Figure 12(*b*) corresponds to the solution resulting from the minimization of the expected value of the residual energy and Fig. 12(*c*) corresponds to the solution resulting from the minimization of the sum of the expected value of the residual energy and variance of the residual energy. The final plot, Fig. 12(*d*) illustrates that minimizing the sum of the expected value of the residual energy and variance of the residual energy and the absolute value of the skew generates a time-delay filter whose performance (worst cost) is comparable to that resulting from the minimax solution.

**5.2 Gaussian Distribution.** Representing the uncertain parameters for a Normally distributed density function as

$$k_1 = k_1^0 H_0(\xi_1) + k_1^1 H_1(\xi_1) \tag{81}$$

and

$$m_3 = m_3^0 H_0(\xi_2) + m_3^1 H_1(\xi_2) \tag{82}$$

where $H_i$ are the Hermite polynomials; we have $k_1^0=1$ and $k_1^1=0.2$ and $m_3^0=2$ and $m_3^1=0.4$. We can use Eq. (79) to represent the system states by replacing the Legendre polynomials $P_i$ with Hermite polynomials. The polynomial chaos expansion can be used to represent the residual energy permitting the calculation of the expected value of the mean, mean+deviation and mean+deviation +absolute value of skew, etc., whose minimization results in the variation of the residual energy illustrated in Figs. 13. Figure 13(*a*) is the minimax solution, which permits comparing the polynomial chaos based solution. It should be noted that although the shape of the residual energy surface in Fig. 13(*d*) is quite different compared with Fig. 13(*a*), the value of the maximum of the surfaces, which corresponds to the minimax cost, are comparable.

## 6 Computational Cost Analysis

In this section, we analyze the computational cost associated with the proposed polynomial chaos based approach and the minimax approach for robust input shaper design. Since both the approaches involve solving an optimization problem, the overall cost can be associated with the computation of the cost function. In the conventional minimax approach, one has to finely sample the uncertain parameter domain to generate a vector of cost. If $n$ is the length of state vector $\mathbf{x}$, $r$ is the length of unknown parameter vector $\mathbf{p}$ and $s$ is the number of samples taken along each dimension, then one must solve $N_{MC}=ns^r$ number of system equations to generate a cost vector for the minimax approach.

For the PC based approach, one has to just solve $n(N+1)$ deterministic equations given by Eq. (21). Here, $(N+1)$ is the number of $r$-dimensional polynomial basis functions involved in PC expansion of each state (Eq. (10)). If $r$-dimensional polynomial

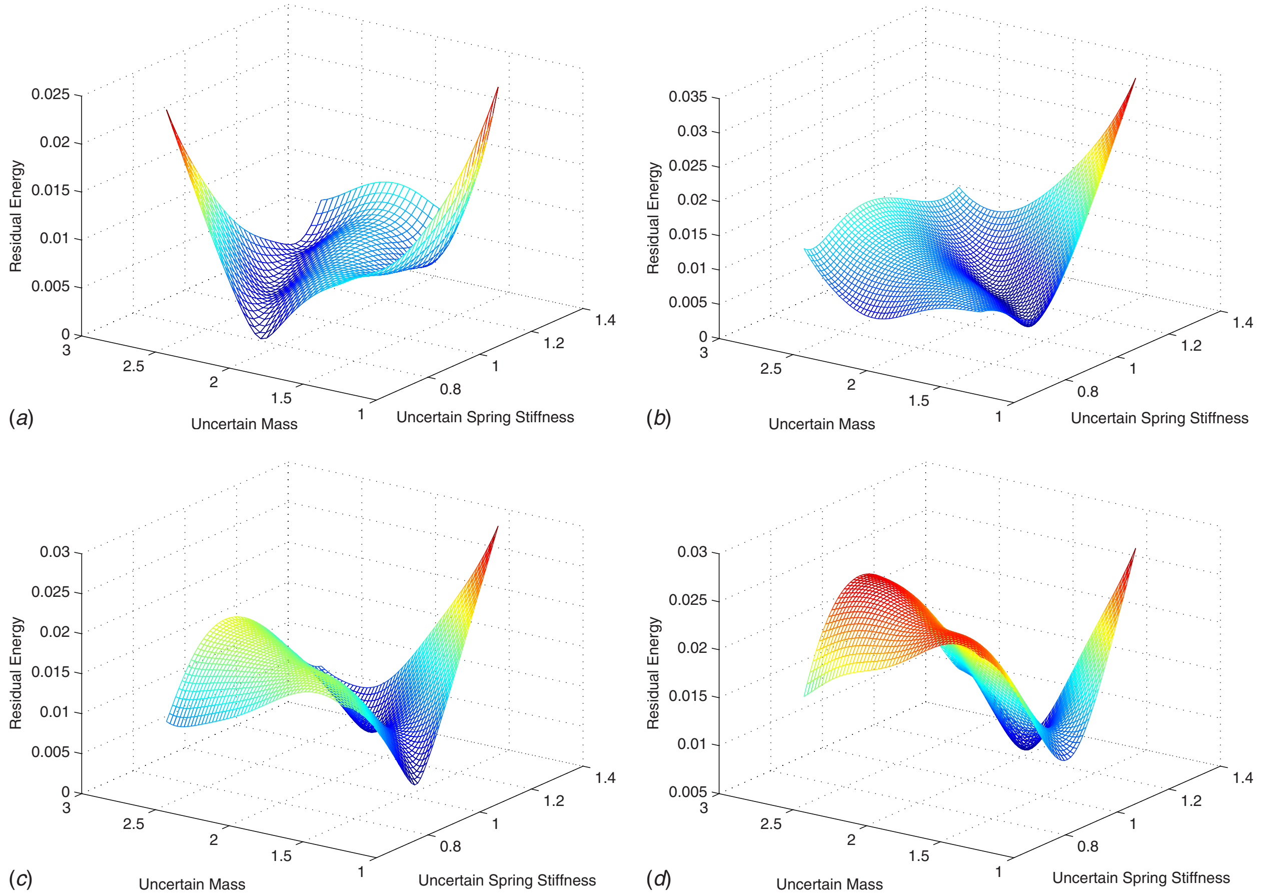



**Fig. 13 Residual energy distribution**

basis functions are generated by tensor product of one-dimensional polynomials, then $N$ is determined by the chosen highest order $l$ of the one-dimensional polynomials and the dimension $(r)$ of the vector of uncertain parameters $\mathbf{p}$ [16].

$$N = \frac{(l+r)!}{l!r!} - 1 \tag{83}$$

Hence, one needs to solve $N_{\mathrm{PC}} = n((l+r)!/l!r!)$ number of system equations to compute the cost associated with proposed approach. As is evident, both the methods suffer from *curse of dimensionality*, i.e., the computational cost associated with both methods grows as the dimension of unknown parameter vector $\mathbf{p}$ increases. Figure 14 illustrates the variation in $N_{\mathrm{MC}}/N_{\mathrm{PC}}$ versus the dimension $(r)$ of the uncertain parameter vector $\mathbf{p}$. The number of samples along each direction in the minimax approach, i.e., $s$ and the order of one-dimensional polynomials $l$ are assumed to be ten although, generally, one require only third to fifth order polynomials for the PC approach. From Fig. 14, it is apparent that the numerical cost associated with the minimax approach increases exponentially as compared with the proposed PC approach.

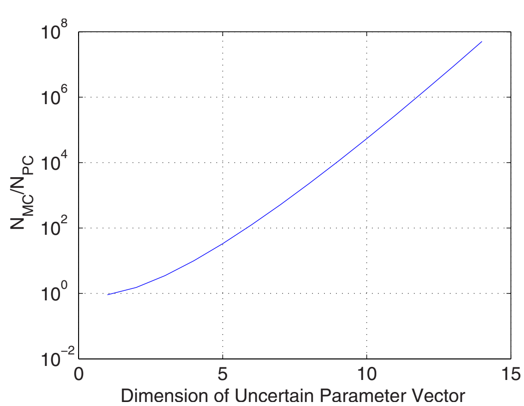



**Fig. 14 Computational cost comparison**

## 7 Conclusions

This paper exploits the polynomial chaos approximation to represent the residual energy random variable. A general development, which permits the use of any probability density function in conjunction with a polynomial chaos based representation of the uncertain variables and the system states is presented. Three probability density function are considered: the first, where the uncertain variable is uniformly distributed, the second, where the distribution is Gaussian, and the third, which is a polynomial function with a compact support. Legendre and Hermite polynomials are used in the polynomial chaos approximation for the Uniform and Gaussian cases, respectively. For the polynomial function with a compact support, the Gram–Schmidt process is used to generate orthogonal functions, which are used in the polynomial chaos expansion. Simulation results for a single spring-mass system illustrate that minimizing the mean and combinations of the higher moments can result in an input shaper, which closely approximates the minimax solution. The benefit of using polynomial chaos approximation to represent the random cost function compared with the minimax approach is the reduction in the computational cost, since one does not need to sample the multidimensional uncertain space, which is required for the minimax approach. A three mass-spring system is used to illustrate how the proposed approach can be easily extended to systems with multiple uncorrelated uncertainties. Minimizing the expected value of the mean, variance and absolute value of the skew is shown to progressively approach the solution of the minimax design.

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