CBRN Data Fusion Using Puff-Based Model and Bar-Reading Sensor Data

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***Abstract*— This article provides an approximate approach to the measurement update of the puff state in the data assimilation process for airborne material dispersion where the sensor measurements are bar readings. Based on the Bayes rule and numerical quadrature techniques, the approach approximates an interval in the concentration space associated with a sensor's bar reading with a set of discrete points and the integrals over the interval by sums of function evaluations at those points. An alternative approximation to the integrals involves the Gaussian error function and the Hermite-Gaussian quadrature. Under the assumption that all the distributions are approximately Gaussian, a two-step procedure of the update is presented: 1) updating the continuous-valued concentration forecast with the bar-reading data and 2) updating the puff state based on the correlation between the puff state and the concentration.**

Keywords: Data assimilation, puff, bar readings, quadrature.

I. Introduction

Lagrangian mesoscale atmospheric dispersion models such as SCIPUFF [1] and RIMPUFF [2] predict the distribution of airborne chemical, biological, radiological, and nuclear (CBRN) materials over time and space. The strength of these models is their simplicity and suitability for real-time tracking of toxic materials. These models represent the spectrum of release mechanisms (instantaneous release to extended time-varying release) by the release of a series of Gaussian-shaped puffs at regular intervals. The consequence of assuming a simple model is the associated loss of model fidelity and increase in bias and uncertainty of the estimate of the concentration distributions. There is thus a need to ameliorate the model forecast errors by exploiting concentration measurements using one or several sensors. This is referred to as data assimilation.

In the dispersion problem, the dominant characteristics of the data assimilation process are the nonlinearity and the high dimensionality of the dispersion forecast model. The computational complexity of an extended Kalman filter or the like for the dispersion problem is high [3]. Monte-Carlo-based methods including the ensemble Kalman filters [4] and the particle filters [5] have also been proposed. All of them have to trade off between performance and complexity.

The puff-based dispersion model involves a time-varying state space due to the change of the number of puffs as a result of adaptive puff splitting and puff merging based on the size and relative distance of the puffs, which prevents the standard use of the ensemble Kalman filters from these applications because the ensemble members of an ensemble Kalman filter may not have the same dimensionality. The particle filters can still be applied because they do not have to rely on inter-particle information [6].

Another challenge of the data assimilation problem for the dispersion process comes from the sensors, which will be addressed in this article. The most useful and widely deployed sensors for measurement of chemical concentration are bar-reading sensors which have coarse resolution and nonlinear transition from one bar, or measurement quantum, to another. For instance, the prototype chemical sensor model in [7] only has eight distinct integer outputs. Any material concentration value within an interval $[T_i, T_{i+1}]$, where T_i and T_{i+1} are two concentration thresholds, is converted into the integer output $\mathbb{I}$ ($\mathbb{I} = i$). In the limiting case, a sensor only has two possible outputs, which only indicate whether the (non-negative) concentration is above or below the detection threshold. Integration of such sensor models is necessary in the synthesis of any useful data assimilation process in this application setting.

The synthesis or measurement update of a Kalman-filter-like method involves solving an inverse problem, that is, updating the parameters in the state space using the data in the output space. Oftentimes, a linear update scheme such as the linear minimum mean square error estimation [8] is used. Not all the particle filters need to solve such an inverse problem. For example, a bootstrap filter propagate the particles through the forecast dynamics and then updates the so-called importance weights on the particles using the likelihood functions of the particles, which are computed based on the forward observation model, and the sensor data. However, incorporating a local measurement update in a particle filter will yield a better importance function and therefore increase the sample efficiency of the particle filter, which is desired especially when the number of particles can only be limited, as in a large-scale dispersion forecast problem.

This article focuses on the measurement update of the puff state with bar-reading sensor data in a recursive estimator. By

use of a recursive estimator, the requirement for data storage and computation complexity is greatly reduced. The objective of the article is to make better use of the information contained in the sensor readings than a naive method in which the effect of quantization in such a sensor is ignored. The proposed measurement update scheme is based on the Bayes rule.

A main assumption of the article is that the functional form of the prior (background) distribution and that of the likelihood function are available. The likelihood function is based on a simplified version of the sensor model from [7], in which the internal noise variable defining the likelihood function is normally distributed. For nonlinear, high-dimensional problems including the dispersion problem under investigation, the prior distribution is usually approximated by an ensemble (as in an ensemble Kalman filter) or a set of particles (as in a particle filter) which are propagated through the full dynamics model. The ensemble- or particle-based propagation method is likely to capture the nonlinearity of the forecast model better than those based on a first-order tangent model. How to reduce the ensemble approximation error and how to convert the ensemble or particles to a presumed analytic form and vice versa are of great importance, and will be investigated in the near future, but are beyond the scope of the article. For the moment, it is assumed that the prior distribution of the puff state and the concentrations is approximately a joint-Gaussian distribution with given mean and covariance. Since a Gaussian distribution is sufficiently represented by its mean and covariance, which can be numerically approximated by the ensemble mean and ensemble covariance, the conversion between the ensemble and the analytic form is trivial if no variable state dimensionality problem is involved or the problem can be circumvented. For mathematical convenience, the posterior distribution of the puff state is also assumed to be normally distributed.

The organization of the article proceeds as follows. First, the dispersion model and the sensor model are briefly reviewed. Next, the problem statement of measurement update is given. Two measurement update methods are then presented followed by the conclusions.

II. Models

In this section, the puff-based dispersion model and the sensor model are reviewed. These models are sometimes referred to as the *forward* models, which are used to derive the prior distribution and the likelihood function of the state vector. In principle, the posterior distribution of the state vector can be obtained from the prior distribution and the likelihood function using the Bayes rule.

A. Gaussian-Puff-Based Dispersion Model

In our application, the dispersion process of a certain type of materials is described by a Lagrangian mesoscale atmospheric dispersion puff model which simulates the time changing release (emission) of airborne materials by sequentially releasing a series of Gaussian shaped puffs at a regular rate on a specified grid. The amount of airborne materials allocated to individual Gaussian puffs equals the release rate times the time elapsed between puff releases. At each time step, the model advects, diffuses and deposits the individual Gaussian puffs according to local meteorological parameter values. Our present model is restricted to the two dimensional horizontal spread of the plume, thereby ignoring the vertical effects on dispersion. A two-dimensional Gaussian distribution function is used to represent a Gaussian puff. This distribution, however, should not be confused with the probability distribution function in the estimation context. For each Gaussian puff, the zeroth moment of the Gaussian distribution represents the amount of materials carried by the puff (if it is a probability distribution, the zeroth moment must be one), the mean represents the location of the puff center, and the covariance parameters represent the size and shape of the puff.

The state vector $\mathbf{x}^{(j)}$ for the j^{th} Gaussian puff consists of six parameters: the mass of the material $Q^{(j)}$, the Cartesian coordinates $x^{(j)}$ and $y^{(j)}$, the standard deviations $\sigma_x^{(j)2}$, $\sigma_y^{(j)2}$, and the correlation $\sigma_{xy}^{(j)}$. That is,

$$\mathbf{x}^{(j)} = [Q^{(j)}, x^{(j)}, y^{(j)}, \sigma_x^{(j)2}, \sigma_y^{(j)2}, \sigma_{xy}^{(j)}]^T \tag{1}$$

Note that $\sigma_x^{(j)}\sigma_y^{(j)} \geq \sigma_{xy}^{(j)}$ because a covariance matrix must be positive semi-definite. A trick to preserve the positive semi-definiteness of the covariance matrix is to propagate and update the square root factor of the matrix. For a puff of nonzero volume, the requirement should be strengthened as $\sigma_x^{(j)}\sigma_y^{(j)} > \sigma_{xy}^{(j)}$. If the puff is assumed circular, the corresponding covariance matrix is parameterized by a single parameter $\sigma^{(j)2} = \sigma_x^{(j)2} = \sigma_y^{(j)2}$. In this case, the number of state variables for a puff reduces to four.

The state vector $\mathbf{x}$ representing all the puffs at any time is the concatenation of $\mathbf{x}^{(j)}$ at the time: $\mathbf{x} = [\mathbf{x}^{(1)T}, \mathbf{x}^{(2)T}, \ldots, \mathbf{x}^{(M)T}]^T$, with M the total number of puffs, which increases with puff release or puff splitting and decreases with puff merging. The size of $\mathbf{x}$ is large when M becomes large, which is a source of the complexity of the data assimilation problem.

Given $\mathbf{x}$, the concentration of the material at a location (x_g, y_g) is computed by

$$\begin{aligned} c \triangleq c(\mathbf{x}; x_g, y_g) = \sum_{j=1}^{M} & \frac{Q^{(j)}}{\sqrt{\det\left(2\pi \cdot P^{(j)}\right)}} \\ & \cdot \exp\left[-\frac{1}{2}(\Delta\mathbf{x}^{(j)})^T P^{-1}(\Delta\mathbf{x}^{(j)})\right] \end{aligned} \tag{2}$$

where $\det$ denotes the determinant of a matrix,

$$\Delta\mathbf{x}^{(j)} = [x^{(j)}, y^{(j)}]^T - [x_g, y_g]^T \tag{3}$$

and

$$P^{(j)} = \begin{bmatrix} \sigma_x^{(j)2} & \sigma_{xy}^{(j)} \\ \sigma_{xy}^{(j)} & \sigma_y^{(j)2} \end{bmatrix} \tag{4}$$

Note that, for the moment, the normal distribution rather than the more realistic clipped normal distribution is used in the above computation of the concentration.

B. Bar-Reading Sensor Model

The sensor model describes the relation between the concentration c and the sensor reading. The outputs of a bar sensor are discrete numbers of bars. In [7], the number of bars ranges from zero to seven. These bar readings indicate the concentration magnitude at the sensor location at the "instant"; the sensor displays $\mathbb{I} = 0, \ldots, 7$, bars when the internal continuous-valued concentration magnitude c_v is between thresholds T_i and T_{i+1}, where $T_{i+1} > T_i \geq 0$. The thresholds are assumed to be exactly known. The properties of the sensor are determined by the thresholds and the properties of c_v, which is assumed normally distributed about the true concentration c. The measurement error $v = c_v - c$ has mean zero and variance

$$R \triangleq R(c) = \alpha c + J \tag{5}$$

where α is the proportionality constant and J accounts for the thermal motion of the electrons in the components [7]. (In practice, because the true value of c is never known, $R(c)$ is usually approximated by $R(c^*)$, where c^* is an estimate of c). The probability density function of c_v given the puff state or the concentration is

$$\begin{aligned} p(c_v|\mathbf{x}) = p(c_v|c) &= \mathcal{N}(c_v; c, R) \\ &= \frac{1}{\sqrt{2\pi R}} \exp\left[-\frac{(c_v - c)^2}{2R}\right] \end{aligned} \tag{6}$$

where $\mathcal{N}(\cdot\,;\cdot\,,\cdot)$ denotes a Gaussian probability density function with the mean and covariance (or variance) specified by the second and third arguments and evaluated at the first argument. Following that, the probability of an output $\mathbb{I}$ conditioned on the concentration c is determined by the following integral

$$P(\mathbb{I}|c) \propto \int_{T_i}^{T_{i+1}} p(c_v|c) dc_v \tag{7}$$

For $P(\mathbb{I}|c)$ to be a probability distribution, we further require

$$\sum_{\mathbb{I}} P(\mathbb{I}|c) = 1 \tag{8}$$

For sensors at different locations, we assume that the internal measurement errors v are independent. That is to say, the internal measurements at different locations are conditional independent

$$\begin{aligned} p(c_v(\mathbf{x}; x_g, y_g), c_v(\mathbf{x}; x'_g, y'_g)|\mathbf{x}) = p(c_v(\mathbf{x}; x_g, y_g)|\mathbf{x}) \\ \cdot p(c_v(\mathbf{x}; x'_g, y'_g)|\mathbf{x}) \end{aligned} \tag{9}$$

Under this conditional independence assumption, the measurement update can be carried out sequentially. Hence, there is no need to process all the sensor data simultaneously, which greatly reduces the computational load (at the price of accuracy degradation due to ignoring the possible correlation between sensors).

III. Problem Statement of Measurement Update

From the Bayesian perspective, the problem of measurement update can be formulated as follows: given the sensor data, the likelihood function, and the prior distribution of the puff state, find out the posterior distribution of the puff state.

Under the Gaussian assumption, finding out the posterior distribution reduces to determining the posterior mean and the associated error covariance of the puff state. Because the sensor measurements have been assumed to be conditional independent, we are allowed to process the sensor data sequentially. Thus, we only need to show how to update the puff state $\mathbf{x}$ with the reading $\mathbb{I}$ of a single sensor. This problem can be divided into two steps: 1) the update of the continuous-valued concentration with the reading $\mathbb{I}$; and 2) the update of the puff state with the updated concentration.

The joint prior distribution of the puff state $\mathbf{x}$ and the concentration c at a certain location is assumed approximately Gaussian:

$$p(\mathbf{x}, c) = \mathcal{N}\left(\begin{bmatrix}\mathbf{x} \\ c\end{bmatrix}; \begin{bmatrix}\hat{\mathbf{x}}^- \\ \hat{c}^-\end{bmatrix}, \begin{bmatrix}P_x^- & P_{xc}^- \\ P_{cx}^- & P_c^-\end{bmatrix}\right) \tag{10}$$

The *a priori* mean and covariance are obtained using the forward model. The marginal distributions are given by

$$p(\mathbf{x}) = \mathcal{N}\left(\mathbf{x}; \hat{\mathbf{x}}^-, P_x^-\right) \tag{11}$$

and

$$p(c) = \mathcal{N}\left(c; \hat{c}, P_c^-\right) \tag{12}$$

Note that because c and c_v can only take on non-negative values, the probability density functions of them are not Gaussian in the strict sense. However, it is assumed that they are well approximated by Gaussian distributions. The likelihood function of the concentration c for the given bar reading $\mathbb{I}$ is given by $P(\mathbb{I}|c)$ as defined in Eq. (7).

IV. Updating the Concentration with the Sensor Reading

A. Gaussian Sum Approximation

Using the Bayes rule, the posterior distribution of c is given by

$$\begin{aligned} p(c|\mathbb{I}) &= \frac{P(\mathbb{I}|c)p(c)}{P(\mathbb{I})} \\ &= \frac{\int_{T_i}^{T_{i+1}} p(c_v|c)p(c)dc_v}{\int_{T_i}^{T_{i+1}} \int p(c_v|c)p(c)dc\, dc_v} \\ &= \frac{\int_{T_i}^{T_{i+1}} p(c_v)p(c|c_v)dc_v}{\int_{T_i}^{T_{i+1}} p(c_v)dc_v} \end{aligned} \tag{13}$$

where

$$p(c_v|c)p(c) = p(c|c_v)p(c_v) \tag{14}$$

and

$$p(c_v) = \int p(c_v|c)p(c)\, dc \tag{15}$$

have been used. Because we assume that all the probability functions are Gaussian, the above integral is defined over

$(-\infty, \infty)$. Because both $p(c_v|c)$ and $p(c)$ are assumed to be Gaussian distributions, the corresponding $p(c|c_v)$ and $p(c_v)$ are also Gaussian. Furthermore, the means and variances of $p(c|c_v)$ and $p(c_v)$ can be easily computed using a Kalman filter. When $p(c_v|c)$ and $p(c)$ are strongly non-Gaussian, there may not be simple ways to compute $p(c|c_v)$ and $p(c_v)$ and Eqs. (14) and (15) would be less useful.

The posterior mean and variance are given by

$$\begin{aligned}\hat{c}^+ &= E_{p(c|\mathbb{I})}[c] \\ &= \int c\, p(c|\mathbb{I}) dc \\ &= \frac{\int_{T_i}^{T_{i+1}} \left[\int c \cdot p(c|c_v) dc\right] p(c_v) dc_v}{\int_{T_i}^{T_{i+1}} p(c_v) dc_v}\end{aligned} \tag{16}$$

and

$$\begin{aligned}P_c^+ &= E_{p(c|\mathbb{I})}\left[(c - E_{p(c|\mathbb{I})}[c])^2\right] \\ &= E_{p(c|\mathbb{I})}[c^2] - \left\{E_{p(c|\mathbb{I})}[c]\right\}^2 \\ &= \int c^2 p(c|\mathbb{I}) dc - (\hat{c}^+)^2 \\ &= \frac{\int_{T_i}^{T_{i+1}} \left[\int c^2 \cdot p(c|c_v) dc\right] p(c_v) dc_v}{\int_{T_i}^{T_{i+1}} p(c_v) dc_v} - (\hat{c}^+)^2\end{aligned} \tag{17}$$

where $E_{p(c|\mathbb{I})}$ denotes the expectation with respect to $p(c|\mathbb{I})$. Define

$$\hat{c}(c_v) = \int c \cdot p(c|c_v) dc \tag{18}$$

$$P_c(c_v) = \int (c - \hat{c}(c_v))^2 \cdot p(c|c_v) dc \tag{19}$$

and

$$\beta(c_v) = \frac{p(c_v)}{\int_{T_i}^{T_{i+1}} p(c_v) dc_v} \tag{20}$$

Obviously,

$$\int_{T_i}^{T_{i+1}} \beta(c_v) dc_v = 1 \tag{21}$$

The posterior mean and variance can now be rewritten as

$$\begin{aligned}\hat{c}^+ &= \frac{\int_{T_i}^{T_{i+1}} \hat{c}(c_v) p(c_v) dc_v}{\int_{T_i}^{T_{i+1}} p(c_v) dc_v} \\ &= \int_{T_i}^{T_{i+1}} \hat{c}(c_v) \beta(c_v) dc_v\end{aligned} \tag{22}$$

and

$$\begin{aligned}P_c^+ &= \frac{\int_{T_i}^{T_{i+1}} (P_c(c_v) + \hat{c}(c_v)^2) p(c_v) dc_v}{\int_{T_i}^{T_{i+1}} p(c_v) dc_v} - (\hat{c}^+)^2 \\ &= \frac{\int_{T_i}^{T_{i+1}} [P_c(c_v) + (\hat{c}(c_v) - \hat{c}^+(c_v))^2] p(c_v) dc_v}{\int_{T_i}^{T_{i+1}} p(c_v) dc_v} \\ &= \int_{T_i}^{T_{i+1}} [P_c(c_v) + (\hat{c}(c_v) - \hat{c}^+)^2] \beta(c_v) dc_v\end{aligned} \tag{23}$$

The above equation shows that additional uncertainty of the concentration estimate comes from the fact that the sensor reading corresponds to an interval rather than a single point.

The integrals over $[T_i, T_{i+1}]$ can be approximated by a summation over a set of selected $c_v^{(l)}$ (they may or may not be equally spaced), that is,

$$\int_{T_i}^{T_{i+1}} p(c_v) p(c|c_v) dc_v \approx \sum_l \alpha_l p(c_v^{(l)}) p(c|c_v^{(l)}) \tag{24}$$

$$\int_{T_i}^{T_{i+1}} p(c_v) dc_v \approx \sum_l \alpha_l p(c_v^{(l)}) \tag{25}$$

with the coefficients α_l satisfying $\sum_l \alpha_l = T_{i+1} - T_i$. We choose to use the same set of α_l and $c_v^{(l)}$ for both integrals. Many numerical quadrature techniques can be applied [9]. Because the integrals are one-dimensional, they can be evaluated accurately at reasonable computational cost. Note that because we have assumed that $p(c_v)$ is a Gaussian distribution, $\int_{T_i}^{T_{i+1}} p(c_v) dc_v$ may also be given by the difference of two error functions, which will be shown in the next subsection.

Substituting the above numerical approximations into Eq. (13) gives

$$\begin{aligned}p(c|\mathbb{I}) &\approx \frac{\sum_l \alpha_j p(c_v^{(l)}) p(c|c_v^{(l)})}{\sum_l \alpha_l p(c_v^{(l)})} \\ &= \sum_l \beta_l p(c|c_v^{(l)})\end{aligned} \tag{26}$$

where

$$\beta_l = \frac{\alpha_l p(c_v^{(l)})}{\sum_l \alpha_l p(c_v^{(l)})} \tag{27}$$

That is, $p(c|\mathbb{I})$ is approximated by a finite sum of Gaussian distributions with β_l the weighting factors. Clearly,

$$\sum_l \beta_l = 1 \tag{28}$$

and scaling α_l by a nonzero constant will not change β_l. The Gaussian distributions $p(c|c_v^{(l)})$ and $p(c_v^{(l)})$ are given by $\mathcal{N}\left(c; \hat{c}^{(l)+}, P_c^{(l)+}\right)$ and $\mathcal{N}\left(c_v^{(l)}; \hat{c}^-, P_c^- + R\right)$, respectively. The posterior estimate $\hat{c}^{(l)+}$ and the associated error variance $P_c^{(l)+}$ are available from standard Kalman filter measurement updates:

$$\hat{c}^{(l)+} = \frac{R \cdot \hat{c}^- + P_c^- \cdot c_v^{(l)}}{P_c^- + R} \tag{29}$$

and

$$P_c^{(l)+} = \frac{P_c^- \cdot R}{P_c^- + R} \tag{30}$$

where R is evaluated at some point in $[T_i, T_{i+1}]$ for all the $c_v^{(l)}$. The mean and variance of $p(c|\mathbb{I})$ are given by

$$\hat{c}^+ = \sum_l \beta_l \hat{c}^{(l)+} \tag{31}$$

and

$$P_c^+ = \sum_l \beta_l \left[P_c^{(l)+} + (\hat{c}^{(l)+} - \hat{c}^+)^2\right] \tag{32}$$

The above equation also indicates that the uncertainty of the concentration estimate would be underestimated if the interval corresponding to a sensor reading is approximated by a single point.

Simple Example: Let us consider a binary sensor that has two possible outputs. The sensor output is 0 when $c \in [0, 1)$ and 1 when $c \in [1, 2)$. The sensor variance is 0.1. The concentration prediction variance is 0.1. The numbers assumed in the example may not be in the correct scale. Two cases are examined.

1) The sensor output is 1; the concentration prediction is 1.0 and the predicted number of bars is 1.
2) The sensor output is 1; the concentration prediction is 1.5 and the predicted number of bars is 1.
3) The sensor output is 1; the concentration prediction is 0.5 and the predicted number of bars is 0.

In the third case, there is no doubt that a measurement update is needed. In the first two cases, because the predicted number of bars is identical to the sensor output, at first sight, the concentration prediction should not be updated. However, according to the Bayes rule, the optimal update is nonzero because of the deviation of the prediction from the "mean" of the interval over $[1, 2)$. Hence, the concentration prediction needs to be updated in both cases.

In a naive method, a single point is chosen to represent the whole interval and this point is used in the Kalman filter measurement update as the measurement data. In this example, the middle of the interval, 1.5, is used. Then, the updated concentration is 1.25 for the first case, 1.5 for the second case, and 1 for the third case. The variance after update is 0.05 for all three cases.

Now, three points, $c_v^{(1)} = 1.0$, $c_v^{(2)} = 1.5$, and $c_v^{(3)} = 2.0$ are chosen for the interval, the weights are $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$ (this choice of equal weights is only for mathematical convenience), and the method in the previous section is used.

In the first case, $\hat{c}^{+(1)} = 1$, $\hat{c}^{+(2)} = 1.25$, and $\hat{c}^{+(1)} = 1.5$; $P^{+(1)} = P^{+(2)} = P^{+(3)} = 0.05$; and $\beta_1 = 0.62$, $\beta_2 = 0.33$, $\beta_3 = 0.05$. Thus, $\hat{c}^{+} = 1.11$ and $P^{+} = 0.07$.

In the second case, $\hat{c}^{+(1)} = 1.25$, $\hat{c}^{+(2)} = 1.5$, and $\hat{c}^{+(1)} = 1.75$; $P^{+(1)} = P^{+(2)} = P^{+(3)} = 0.05$; and $\beta_1 = 0.26$, $\beta_2 = 0.48$, $\beta_3 = 0.26$. Thus, $\hat{c}^{+} = 1.5$ and $P^{+} = 0.08$.

In the third case, $\hat{c}^{+(1)} = 0.75$, $\hat{c}^{+(2)} = 1.0$, and $\hat{c}^{+(1)} = 1.25$; $P^{+(1)} = P^{+(2)} = P^{+(3)} = 0.05$; and $\beta_1 = 0.86$, $\beta_2 = 0.13$, $\beta_3 = 0.01$. Thus, $\hat{c}^{+} = 0.79$ and $P^{+} = 0.06$.

The difference between the two methods is clear in this example.

B. Alternative Approximation

The Gaussian sum approximation is based on expressing the posterior distribution in terms of $p(c_v)$ and $p(c|c_v)$ (see Eq. (13)) and the assumption that $p(c_v)$ and $p(c|c_v)$ are Gaussian distributions. An alternative approximation scheme using $p(c_v|c)$ and $p(c)$ directly is considered in this subsection. Other similar approximations may be possible as well.

Define

$$I_i(c) = \int_{T_i}^{T_{i+1}} p(c_v|c)dc_v \tag{33}$$

which is identical to $P(\mathbb{I}|c)$ up to a normalizing constant. Under the Gaussian assumption for $p(c_v|c)$, $I_i(c)$ can be calculated as the difference of two error functions or two complementary error functions. An error function for a standard Gaussian distribution with zero mean and unity variance is defined as [9]

$$\text{erf}(c) = \frac{2}{\sqrt{\pi}} \int_0^c \exp(-t^2)dt \tag{34}$$

A complementary error function for standard Gaussian distribution with zero mean and unity variance is defined as [9]

$$\text{cerf}(c) = \frac{2}{\sqrt{\pi}} \int_c^\infty \exp(-t^2)dt \tag{35}$$

Hence,

$$\begin{aligned} I_i(c) &= \frac{1}{\sqrt{2\pi R}} \int_{T_i}^{T_{i+1}} \exp\left[-\frac{(c_v - c)^2}{2R}\right] dc_v \\ &= \frac{1}{2}\left[\text{erf}\left(\frac{T_{i+1} - c}{\sqrt{2R}}\right) - \text{erf}\left(\frac{T_i - c}{\sqrt{2R}}\right)\right] \\ &= \frac{1}{2}\left[\text{cerf}\left(\frac{T_i - c}{\sqrt{2R}}\right) - \text{cerf}\left(\frac{T_{i+1} - c}{\sqrt{2R}}\right)\right] \end{aligned} \tag{36}$$

In terms of $I_i(c)$, the posterior distribution can be rewritten as

$$\begin{aligned} p(c|\mathbb{I}) &= \frac{p(\mathbb{I}|c)p(c)}{p(\mathbb{I})} \\ &= \frac{I_i(c)p(c)}{\int I_i(c)p(c)dc} \end{aligned} \tag{37}$$

Accordingly, the posterior mean and variance of $p(c|\mathbb{I})$ are

$$\hat{c}^{+} = \frac{\int cI_i(c)p(c)dc}{\int I_i(c)p(c)dc} \tag{38}$$

and

$$P_c^{+} = \frac{\int c^2 I_i(c)p(c)dc}{\int I_i(c)p(c)dc} - (\hat{c}^{+})^2 \tag{39}$$

Noting that $p(c)$ is a Gaussian distribution, we will use the Hermite-Gaussian quadrature method [9] to approximate the above integrals.

The basic idea of the Hermite-Gaussian quadrature method is to approximate

$$\int_{-\infty}^{\infty} f(c) \exp(-c^2)dc$$

by

$$\sum_{l=1}^{L} \alpha_l f(c_l)$$

The points c_l are the L roots of the Hermite polynomial of degree L. The coefficients α_l are the associated weights on the points. The Hermite polynomials are defined by [9]

$$H_L(x) = (-1)^n e^{x^2} \frac{d^L}{dx^L} e^{-x^2} \tag{40}$$

For example,

$$\begin{aligned} H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \\ H_4(x) &= 14x^4 - 48x^2 + 12 \end{aligned} \tag{41}$$

The associated weights are given by [9]

$$\alpha_l = \frac{2^{L-1} L! \sqrt{\pi}}{L^2 [H_{L-1}(x_l)]^2} \tag{42}$$

with $L!$ the factorial of L. It has been known that close relation exists between the Hermite-Gaussian quadrature and the Unscented Transformation [10]. See [11] and [12], for example.

Given c_l and α_l, we have the following approximations:

$$\int I_i(c) p(c) dc = \sum_l \alpha_l I_i(c_l) \tag{43}$$

$$\int c I_i(c) p(c) dc = \sum_l \alpha_l c_l I_i(c_l) \tag{44}$$

and

$$\int c^2 I_i(c) p(c) dc = \sum_l \alpha_l c_l^2 I_i(c_l) \tag{45}$$

Following that, we have

$$\hat{c}^+ = \frac{\sum_l \alpha_l c_l I_i(c_l)}{\sum_l \alpha_l I_i(c_l)} \tag{46}$$

and

$$P_c^+ = \frac{\sum_l \alpha_l c_l^2 I_i(c_l)}{\sum_l \alpha_l I_i(c_l)} - (\hat{c}^+)^2 \tag{47}$$

Higher moments can be computed similarly. The higher moments may be informative about how well the underlying distribution $p(c|\mathbb{I})$ is approximated by a single Gaussian distribution. The approximation method in this subsection may be more easily modified than the method in the previous section in order to apply to cases in which $p(c)$ and $p(c_v|c)$ are non-Gaussian.

V. Updating Puff State with Updated Concentration

In the previous section, efforts were made to compute $\hat{c}^+$ and P_c^+ accurately in the presence of bar readings. The update of the puff state is addressed in this section. Given the large size of the puff state vector and the high computational complexity associated with high-order measurement update, only a correlation-based linear update scheme is used to update the puff state. We assume that the posterior puff distribution $p(\mathbf{x}|\mathbb{I})$ is approximated by a Gaussian distribution, denoted by $\mathcal{N}(\mathbf{x}; \hat{\mathbf{x}}^+, P_{xx}^+)$. Given $\hat{c}^+$ and P_c^+, the puff state and its associated error covariance can be updated as follows:

$$\hat{\mathbf{x}}^+ = \hat{\mathbf{x}}^- + \mathcal{K}(\hat{c}^+ - \hat{c}^-) \tag{48}$$

$$P_x^+ = P_x^- + \mathcal{K}(P_c^+ - P_c^-)\mathcal{K}^T \tag{49}$$

with

$$\mathcal{K} = P_{xc}^- (P_{cc}^-)^{-1} \tag{50}$$

If $p(c|\mathbb{I})$ cannot be well approximated by a single Gaussian distribution with mean $\hat{c}^+$ and error covariance P_c^+, for example, when the interval $[T_i, T_{i+1}]$ is large, the above update method may not yield the desired accuracy.

Suppose $p(c|\mathbb{I})$ is much better approximated by a Gaussian sum, given by Eq. (26), than by $\mathcal{N}(\mathbf{x}; \hat{\mathbf{x}}^+, P_{xx}^+)$. The posterior distribution of the puff state in this case is also a Gaussian sum. The updated mean and covariance for each Gaussian component are given by

$$\hat{\mathbf{x}}^{(l)+} = \hat{\mathbf{x}}^- + \mathcal{K}(\hat{c}^{(l)+} - \hat{c}^-) \tag{51}$$

$$P_x^{(l)+} = P_{xx}^- + \mathcal{K}(P_c^{(l)+} - P_c^-)\mathcal{K}^T \tag{52}$$

Finally, the mean and covariance for $p(\mathbf{x}|\mathbb{I})$ are

$$\hat{\mathbf{x}}^+ = \sum_l \beta_l \hat{\mathbf{x}}^{(l)+} \tag{53}$$

and

$$P_x^+ = \sum_l \beta_l \left\{ P_x^{(l)+} + \left[\hat{\mathbf{x}}^{(l)+} - \hat{\mathbf{x}}^+\right] \left[\hat{\mathbf{x}}^{(l)+} - \hat{\mathbf{x}}^+\right]^T \right\} \tag{54}$$

To validate the measurement update method presented in this article, the method is integrated in an extended Kalman filter and tested in a simulated dispersion process with a puff-based dispersion model and a bar-reading sensor model. The estimation accuracy improvement using this extended Kalman filter over the model forecast is shown in Figure 1. The marginal improvement by this extended Kalman filter may be due to the drawbacks of the extended Kalman filtering theory, which assumes the validity of the first-order approximations for the errors and noise. However, it is worth pointing out that the performance of this extended Kalman filter is better than that of an extended Kalman filter with the measurement update using a single point in the concentration interval specified by the two consecutive thresholds corresponding to the sensor reading. Integrating this measurement update technique in the design of the importance function of a particle filter is expected to increase the efficiency of the particle filter.

VI. Conclusions

Data fusion for casualty mitigation in CBRN incidents caused either accidentally or intentionally begins with the assimilation of data from atmospheric sensors measuring concentrations of dangerous materials. Most frequently these sensors have bar-reading outputs, and the nonlinearities introduced can significantly degrade their use in correcting model-based predictions of plume dispersion. Two approximate methods are proposed to address the measurement update problem associated with the bar sensor based on the Gaussian distribution assumptions and the Bayes rule. The methods are useful for improving the data fusion performance of the Kalman filters as well as the particle filters. The comparative study of the two update methods, the development of methods for non-Gaussian $p(c)$ and $p(c_v|c)$, for example, $p(c)$ and

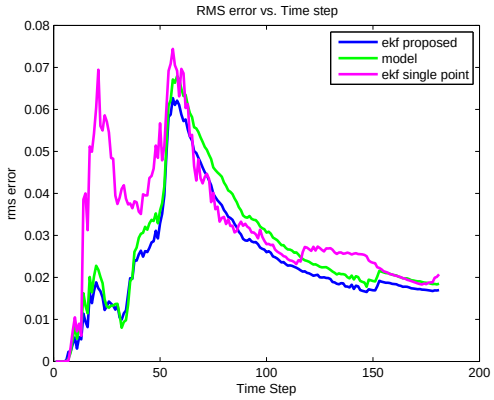

Fig. 1. RMS Errors

$p(c_v|c)$ are log-normal distributions, and the integration of the method in a particle filter are under investigation.

ACKNOWLEDGMENT

This work was supported by the Defense Threat Reduction Agency (DTRA) under Contract No. W911NF-06-C-0162. The authors gratefully acknowledge the support and constructive suggestions of Dr. John Hannan of DTRA.

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