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Convergence and Stability in Distributed Design of Large Systems

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ABSTRACT

Decentralized systems constitute a special class of design under distributed environments. They are characterized as large and complex systems divided into several smaller entities that have autonomy in local optimization and decision-making. The mechanisms behind this network of decentralized design decisions create difficult management and coordination issues. Standard techniques to modeling and solving decentralized design problems typically fail to understand the underlying dynamics of the decentralized processes and therefore result in suboptimal solutions. This paper aims to model and understand the mechanisms and dynamics behind a decentralized set of decisions within a complex design process. This paper builds on already existing results of convergence in decentralized design for simple problems to extend them to any kind of quadratic decentralized system. This involves two major steps: developing the convergence conditions for the distributed optimization problem, and finding the equilibrium points of the design space. Illustrations of the results are given in the form of hypothetical decentralized examples.

Keywords: Decentralized Design, Process Convergence, Decomposition, Game Theory, Nash Equilibrium, Linear System Theory.

INTRODUCTION

The focus of this paper is a theoretical study of the design of complex engineering systems, or those systems that necessitate the decomposition of the system into smaller subsystems in order to reduce the complexity of the design problems. Most of these systems are very large and multidisciplinary in nature, and therefore have a great number of subsystems and components. This creates issues in understanding the interactions between all these subsystems, in order to create more efficient design processes. In this paper, we focus on the dynamics of distributed design processes and attempt to understand the fundamental mechanics behind these processes in order to facilitate the decision process between networks of decision makers.

The novelty of this paper stands in the fact that it extends the results published in [1]. The latter paper had presented preliminary results for convergence in decentralized design problems. It lays out the first steps towards the study of stability for a certain class of design problems with quadratic objective functions: problems involving two designers or design teams, each controlling only one design variable. This paper goes further by giving convergence conditions for *any* kind of quadratic decentralized optimization problem. In this first section, we describe the main properties of complex distributed systems, which are the focus of this paper.

The multidisciplinary nature of these systems make it impossible for one designer, or even a single design team, to consider the entire system as a single design problem. Typically, in complex systems, breaking it up into smaller units or *subsystems* will make the system more manageable [2, 3].

The decentralization of decisions is unavoidable in a large organization where having only one centralized decision maker is usually not applicable [4]. A more effective way is to delegate decision responsibilities to the appropriate person, team or supplier. In fact, decentralization is recommended as a way to speed up product development processes and decrease the computational time and the complexity of the problem [5].

While the decomposition of complex problems certainly creates a series of smaller, less complex problems, it also creates several challenging issues associated with the coordination of these less complex problems. The origin of these problems is the fact that the less complex subproblems are usually coupled and dependent upon information from other subproblems. The ideal case would be when a system could be broken up into subsystems without interdependence. Unfortunately, there are usually design variables and parameters that have an influence on several subproblems. A formal definition of coupled subsystems can be found in [6].

Previous work has been done on the decomposition of the system into smaller ones; using Design Structure Matrices [7], a hierarchical approach [8], or by effectively propagating the desirable top level design specifications to appropriate subsystems [9], and their efficiency has also been compared [10].

Also previous work has concentrated on solving those design problems with interacting subsystems using Game Theory. The main goal is to try to improve the quality of the final solution in a multiobjective, distributed design optimization problem [11]. Previous work in Game Theory includes work to model the interactions between the designers if several design variables are shared among designers [12]. In [13], Game Theory is formally presented as a method to help designers make strategic decisions in a scientific way. In [14], distributed collaborative design is viewed as a non-cooperative game, and maintenance considerations are introduced into a design problem using concepts from Game Theory. In [15], the manufacturability of multi-agent process planning systems is studied using Game Theory concepts. In [16], non-cooperative protocols are studied and the application of Stackelberg leader/follower solutions is shown. Also in [17], a Game Theory approach is used to address and describe a multifunctional team approach for concurrent parametric design. This set of previous work has established a solid foundation for the application of game theory in design, but has not directly studied the mechanisms of convergence in a generic decentralized design problem.

This paper does not propose any other decomposition method, nor another Game-Theoretic approach to the design process. However, it tries to formally describe the dynamics and interactions involved in such design scenarios. We believe that, in order to be able to design better, those dynamics have to be well understood. They will be a strong basis for further research in this area. As Tufte puts it,

An essential analytic task in making decisions based on evidence is to understand how things work - mechanism, trade-offs, process and dynamics, cause and effect. That is, intervention-thinking and policy-thinking demand causality-thinking [18].

Therefore, explaining and understanding the dynamics involved will help us make better decisions in design, and it is the goal of this paper. The next section presents the background for this work, in terms of problem formulation for decentralized decision processes.

DESIGN SCENARIOS

In this section, the main game theory scenarios used to solve large multiobjective design problems are reviewed and discussed. We assume that the design problem has already been subdivided into smaller subsystems, either naturally because several different companies interact on the design of the same product, or either because the system has been subdivided into smaller subsystems using one of the techniques described in the previous section. A good description of the different scenarios in design can be found in [16] and [19].

As mentioned in the previous section, Game Theory is usually used as a way to study those design scenarios. Table 1 presents the Game-Theoretic formulation for an optimization design problem with two designers (also called *players*). In this table, $\mathbf{x}_1$ represents the vector of design variables controlled by designer 1, while designer 2 controls design variable vector $\mathbf{x}_2$. We denote $\mathbf{x}_{1c}$ and $\mathbf{x}_{2c}$ the *nonlocal* design variables, variables that appear in a model but are controlled by the other player. In some decomposed problems, one variable may be local to many subsystems. This kind of problem is not investigated in this paper, but is part of the current work of our research.

A complete description of all the protocols can be found in [20], but we present here only the three main types.

Cooperative Protocol

In this protocol, both players have knowledge of the other player's information and they work together to find a Pareto solution. A pair ($\mathbf{x}_{1P}$, $\mathbf{x}_{2P}$) is Pareto optimal [21] if no other pair ($\mathbf{x}_1$, $\mathbf{x}_2$) exists such that

$$\begin{aligned} & F_i(\mathbf{x}_1,\mathbf{x}_2) \leq F_i(\mathbf{x}_{1p},\mathbf{x}_{2p}) \quad i=1,2 \\ \& \quad & F_j(\mathbf{x}_1,\mathbf{x}_2) < F_j(\mathbf{x}_{1p},\mathbf{x}_{2p}) \quad \text{for at least one } j=1,2 \end{aligned} \tag{1}$$

Systems thinking is the key to full cooperation in modern organizations where a shared vision is common and subscribed to by all members of an organization [22]. However, shared vision does not suggest that the designers will necessarily fully cooperate. Mathematical and model cooperation are required to assume full cooperation and that the final design will be Pareto optimal. Unfortunately, this is rarely the case in distributed environments, as there are several obstacles to this full cooperation.

Player 1's Model:	**Player 2's Model**
Minimize	*Minimize*
$\mathbf{F}_1(\mathbf{x}_1, \mathbf{x}_{2c}) = \{F_1^1, F_1^2, ..., F_1^p\}$	$\mathbf{F}_2(\mathbf{x}_2, \mathbf{x}_{1c}) = \{F_2^1, F_2^2, ..., F_2^q\}$
subject to	*subject to*
$g_j^1(\mathbf{x}_1, \mathbf{x}_{2c}) \leq 0 \quad j = 1..m_1$	$g_j^2(\mathbf{x}_2, \mathbf{x}_{1c}) \leq 0 \quad j = 1..m_2$
$h_k^1(\mathbf{x}_1, \mathbf{x}_{2c}) = 0 \quad k = 1..l_1$	$h_k^2(\mathbf{x}_2, \mathbf{x}_{1c}) = 0 \quad k = 1..l_2$
$\mathbf{x}_{1L} \leq \mathbf{x}_1 \leq \mathbf{x}_{1U}$	$\mathbf{x}_{2L} \leq \mathbf{x}_2 \leq \mathbf{x}_{2U}$

Table 1. MULTI-PLAYER OPTIMIZATION PROBLEM FORMULATION

Noncooperative Protocol

This protocol occurs when full coalition among players is not possible due to organizational, information, or process barriers. Players must make decisions by assuming the choices of the other decision makers. In an iterative approach, the final solution would be a Nash equilibrium. A strategy pair ($\mathbf{x}_{1N}$, $\mathbf{x}_{2N}$) is a Nash solution if

$$\begin{aligned} & F_1(x_{1N}, x_{2N}) = \min_{x_1} F_1(x_1, x_{2N}) \\ \text{and} \quad & F_2(x_{1N}, x_{2N}) = \min_{x_2} F_2(x_{1N}, x_2) \end{aligned} \tag{2}$$

In other words, *A point is said to be a Nash Equilibrium or a Nash Solution if no designer can improve unilaterally his/her objective function* [23]. This solution has the property of being individually stable, but is not necessarily collectively optimal, meaning that, at this point, each designer will perceive the design point to be optimal [24], whereas the solution is not necessarily Pareto optimal. This is because any unilateral decision to change a design variable value by either designer can not, by definition, result in a better objective function value for the designer who makes the change. The Nash Equilibrium also has the property of being the fixed point of two subsets of the feasible space:

$$(x_{1N}, x_{2N}) \in X_{1N}(x_{2N}) \times X_{2N}(x_{1N})$$

$$\begin{aligned} \text{where} \quad & X_{1N}(x_2) = \{x_{1N} \,|\, F_1(x_{1N}, x_2) = \min_{x_1} F_1(x_1, x_2)\} \\ & X_{2N}(x_1) = \{x_{2N} \,|\, F_2(x_1, x_{2N}) = \min_{x_2} F_2(x_1, x_2)\} \end{aligned}$$

are called the *Rational Reaction Sets* of the two players. The Rational Reaction Set (RRS) of a player is a function that embodies his reactions to decisions made by other players.

Leader/Follower Protocol

When one player makes their decision first, they have a leader/follower relationship [25]. This is a common occurrence in a design process when one discipline plays a large role early in the design, or in a design process that involves a sequential execution of interrelated disciplinary processes. Player 1 is said to be the leader is he/she declares his/her strategy first, by assuming that Player 2 behaves rationally. Thus the model of Player 1 as a leader is the following

$$\begin{aligned} \textit{Minimize} \quad & F_1(x_1, x_2) \\ \textit{subject to} \quad & x_2 \in X_{2N}(x_1) \end{aligned} \tag{3}$$

where $X_{2N}(x_1)$ is the Rational Reaction Set of Player 2.

The next section explains how these scenarios apply to an engineering design process, and explains the concepts of equilibriums and stability, and their implications on the design process.

EQUILIBRIUM AND STABILITY OF THE DESIGN SPACE

Once again, the focus of this paper is the design of engineering products in decentralized environments. In that case, even within the same corporation, perfect information and cooperation is difficult to achieve due to several factors, including the complexity of the design, geographic separation or information privacy. Therefore, we focus on noncooperative relationships between designers. In other words, we focus on decentralized design scenarios where full and efficient exchange of all information among subsystems is not possible.

Even though most companies are trying to break down the walls between the different disciplines, many decisions are taken in a sequential manner. We are not suggesting here that designers and companies should not strive for cooperation, but that noncooperation is an involuntary result of organizational or informational barriers among decision makers. In particular, competitive suppliers designing parts for the same overall product are usually not willing to share their analysis models, thus also resulting in noncooperation.

The presence of non-local variables in the model of subsystems requires a certain level of communication between the design teams. In a sequential approach, for example, this information flow goes back and forth between the design teams until they reach an agreement on a particular design point. This point is known as a Nash equilibrium, whose properties are shown in Equation (2). The fact that the designers agree on a final design is known as *convergence* of the design process [1]. The issue of divergence in an engineering design process was noted as early as in [11], and remains an issue to be solved [13]. What happens in those cases is that the sequential approach taken by the designers is endless [26]. Exchanging design variables values back and forth, the design teams cannot agree on a final design because, at each iteration, at least one designer will not be satisfied by the point chosen. Figure 1 shows a simple decentralized example involving two designers, each controlling one design variable [1, 11]. Starting with the initial design $(x = 0, y = 0.8)$, it shows the iteration of the designers between their own Rational Reaction Set, and it is obvious to see that it results in a divergent process where designers will not agree on a final design.

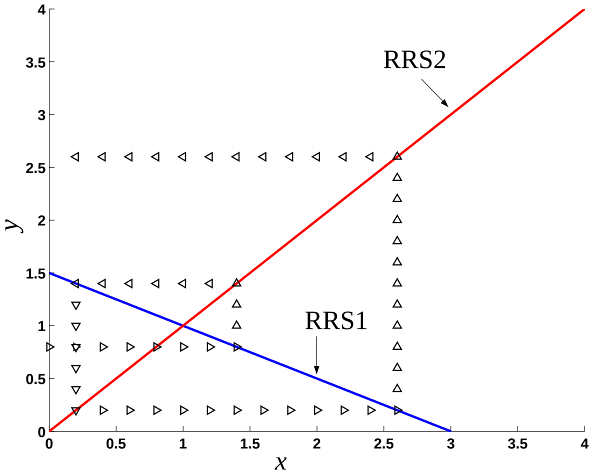

Figure 1. A DIVERGENT DECENTRALIZED DESIGN EXAMPLE

This issue of an unstable equilibrium is challenging. Indeed, in the case of divergence, designers will never agree on a final design since one of the designers will always be able to change the value of their design variables and improve their objective function. In this case, the process by which the two designers might go about choosing the final design is then difficult to predict, but in the absence of any additional information or intervention by a third party, it seems obvious that choosing the final design will be problematic. The first steps towards the study of stability have been laid out for quadratic problems involving two designers, each controlling only one design variable [1]. The focus of this paper is to study those same properties for *any kind* of quadratic optimization problem. This is the novelty of this paper and it lies on two main steps: extending the results for *any number of designers*, and also *any number of design variables* for each designer. The next section presents our approach for solving the issues of stability of the design space.

STABILITY ANALYSIS APPROACH

This section focuses on explaining the basic approach to study the stability of a decentralized decision problem. As mentioned earlier, we only study *quadratic* distributed problems in this paper. By quadratic, we mean design problems with quadratic objective functions and linear constraints. Indeed, constraints can be included in the objective function to create an unconstrained problem by the use of a penalty function [27] and many algorithms use this approach, including many genetic algorithms, simulated annealing, and other heuristic method applications. Moreover, many optimization techniques and algorithms use simplifications or approximations. Some concepts have been developed, that allow the use of multiple approximation models of different types simultaneously in one optimization problem [28]. In addition, quadratic approximations are also used in optimization often since they are not as trivial as linear approximations but are not too complex either [29–32]. The most commonly used approximating functions are polynomial response surface equations [33]. Therefore, studying the behavior of quadratic distributed problems will provide us with a good insight on the dynamics involved in a wide range of design problems. Besides, most papers study the behavior of decentralized decision problems involving two or three design teams, while most complex engineering design problems require a decomposition in a larger number of subsystems. This paper tackles this issue by presenting methods that can be used for design problems with *any number* of designers, each of them controlling *any number* of design variables. The following presents the main steps of our approach.

1. Find the Rational Reaction Sets

The first step for analyzing the stability properties of a design process is to find the equilibrium points of the design space. As mentioned earlier, they lie at the intersection of m subsets of the design space, the Rational Reaction Sets, where m is the number of designers or design teams involved in the design process. We denote n_i the number of design variables controlled by designer i. We also denote $\mathbf{x}$ the *state vector* or vector of all the design variables, grouping all the design variables of every designer, while $\mathbf{x}_i$ is the design vector associated to designer i. Therefore, the length of $\mathbf{x}$ is defined as

$$N = \sum_{i=1}^{m} n_i \tag{4}$$

Finding the Rational Reaction Sets in those conditions is done by setting to zero the first partial derivative of the pseudo-objective function obtained by adding the penalty term (due to the constraints) to the objective function of each designer. Indeed, using a penalty term for the constraints makes the optimization problem unconstrained, making it easy to find the global minimum of an objective function in terms of the other designers' design variables by simply setting the first partial derivatives to zero. Practically, this is done by holding constant the design variables controlled by all the other designers and taking the partial derivative with respect to the design variables he or she is controlling (to study the influence of changing their values). Therefore, the equation of the Rational Reaction Set of designer i is shown in Equation (5) where $\mathbf{F}_i$ is the pseudo-objective function of designer i.

$$\mathrm{RRS}_i \qquad \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_i} = 0 \tag{5}$$

Finding the Rational Reaction Set for every designer will therefore provide us with m sets of equations representing the rational behavior of every designer. Each set is a vector of n_i scalar equations. They give the values of the design variables of a designer at an iteration, as a function of the values of the design variables of the other designers at the previous iteration. They can also be rewritten as N scalar equations, one for each design variable.

2. Find the equilibrium points

The equilibrium points lie at the intersection of the Rational Reaction Sets of every designer. This can be calculated using the set of N equations defined by Equation (5). Since we are considering quadratic problem in this paper, these N equations will be linear, because they are obtained by taking the first derivative of the quadratic pseudo-objective function. Therefore, to find the equilibrium points of the design space, we need to solve a system of N linear equations with N unknowns (the design variables). This system has either no solution (meaning that there is no Nash equilibrium), an infinite number of solutions (a line of Nash equilibriums for example), or a unique solution. An infinite number of Nash solutions is unlikely, because it would require every designer to have the same RRS in some region of the design space. Therefore a quadratic distributed optimization problem will primarily have either one Nash equilibrium or none. This (potential) Nash equilibrium point is the only final design attainable by distributed designers using a sequential approach, but it does not necessarily mean that the designers will converge to it. It depends on the stability of this equilibrium, which is the point of study of this paper, and which is the next logical step in our approach.

3. Study the stability of the equilibrium

Similarly to the notion of equilibriums in physics, equilibrium points in the design space of an engineering design problem can be either stable or unstable. A quadratic distributed decision making problem is defined as a stable system if, independent of the values of the initial conditions, it goes to a steady state in a finite time [34]. In our quadratic environment, the steady state point would naturally be the Nash equilibrium found at the previous step.

Studying the stability of the equilibrium using concepts adapted from Control Theory is the main point of this paper and the details are explained in the next section. First, we present the existing results in this area.

Existing results

This issue of convergence of quadratic decentralized decision systems have already been studied for some given particular cases. Equilibrium points and conditions for convergence have been found for quadratic distributed problems with *two* designers each controlling *one* design variable [1], and with *two* designers each controlling *any number* of design variables [35].

We here present a summary of the results presented in [35], as the notations are used later in the development of the paper. The form of the objective functions for the two designers is a general quadratic equation shown in Equation (6).

$$\begin{aligned} F_1 &= \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{y}^T\mathbf{B}\mathbf{y} + \mathbf{y}^T\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{x} + \mathbf{E}\mathbf{y} + \mathbf{F} \\ F_2 &= \mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{x}^T\mathbf{R}\mathbf{x} + \mathbf{x}^T\mathbf{S}\mathbf{y} + \mathbf{T}\mathbf{y} + \mathbf{U}\mathbf{x} + \mathbf{V} \end{aligned} \tag{6}$$

with designer 1 choosing $\mathbf{x}$ and designer 2 choosing $\mathbf{y}$.

Since the designers are controlling more than one design variable, the control variables $\mathbf{x}$ and $\mathbf{y}$ are design variable vectors. Similarly, $\mathbf{A}$, $\mathbf{B}$ ... are matrices whose sizes can be determined depending on the size of $\mathbf{x}$ and $\mathbf{y}$.

From these formulations, the equations of the Rational Reaction Sets of both designers can be found. Since it is a sequential approach and since the values of the design variables at one iteration are function of the values of the other design variables at the previous iteration, the Rational Reaction Sets are expressed as time series. A matrix formulation of these time series is shown in Equation (7).

$$\begin{aligned} \mathbf{x}_{n+1} &= -\frac{1}{2}\mathbf{A}^{-1}\mathbf{C}^T\mathbf{y}_n - \frac{1}{2}\mathbf{A}^{-1}\mathbf{D}^T \\ \mathbf{y}_{n+1} &= -\frac{1}{2}\mathbf{Q}^{-1}\mathbf{S}^T\mathbf{x}_n - \frac{1}{2}\mathbf{Q}^{-1}\mathbf{T}^T \end{aligned} \tag{7}$$

Next, the Characteristic Matrix of this two-designer decentralized decision problem with quadratic objective functions is defined in Equation (8) and is denoted $\mathbf{K}$.

$$\mathbf{K} = \frac{1}{4}\mathbf{A}^{-1}\mathbf{C}^T\mathbf{Q}^{-1}\mathbf{S}^T \tag{8}$$

Using matrix series, the convergence of the design process can then be studied. It is shown that the stability of the equilibrium point of the design process depends on the eigenvalues of the characteristic matrix $\mathbf{K}$. More particularly, the convergence criterion for the design process can be formulated as shown in Equation (9).

$$\text{The design process converges iff} \quad r_\sigma(\mathbf{K}) < 1 \tag{9}$$

where $r_\sigma(\mathbf{K})$ is the spectral radius of the matrix $\mathbf{K}$ and is defined in Equation (10).

$$r_\sigma(\mathbf{K}) = \max\{|\lambda| : \lambda = \text{eigenvalue of } \mathbf{K}\} \tag{10}$$

While this two-designer scenario can be solved using matrix series, it cannot be applied to more complex design problems involving a great number of designers, thus constitutes the limitations of this method. A more general way of solving the stability of equilibriums of quadratic distributed is presented in the next section, using some of the notation introduced in this section.

DEVELOPMENT OF THE NEW METHOD

Linear System Theory is used in this section as a tool to solve the particular of problems of interest in this paper: quadratic distributed optimization problems with m designers, designer i controlling n_i design variables for a total number of N design variables. $\mathbf{x}$ represents the vector grouping all the design variables of every designer.

Linear System Theory analyses mathematical description of physical systems. Similarly, in this paper, we describe mathematically the interactions of designers acting in a distributed environment, our *physical system*. Linear System Theory concentrates on quantitative analysis (where the responses of systems excited by certain inputs are studied), and on qualitative analysis (which investigates the general properties of systems, such as stability). Qualitative analysis is very important, because design techniques may often evolve from this study [36]. This paper proposes a qualitative analysis of distributed problems, and further analogies with Linear System Theory are made later on.

The mathematical representation used in this paper is similar to the one used in [1, 35]. The pseudo-objective function for designer i is shown in Equation (11).

$$F_i = \mathbf{x}_i^T\,\mathbf{A}_i\,\mathbf{x}_i + \mathbf{x}_{-i}^T\,\mathbf{B}_i\,\mathbf{x}_{-i} + \mathbf{x}_{-i}^T\,\mathbf{C}_i\,\mathbf{x}_i + \mathbf{D}_i\,\mathbf{x}_i + \mathbf{E}_i\,\mathbf{x}_{-i} + \mathbf{F}_i \tag{11}$$

where $\mathbf{x}_i$ denotes the vector of design variables controlled by designer i, and $\mathbf{x}_{-i}$ the vector of design variables *not* controlled by designer i: $\mathbf{x}_{-i} = \mathbf{x} \setminus \mathbf{x}_i = \{x_j \in \mathbf{x}, x_j \notin \mathbf{x}_i\}$.

The matrix $\mathbf{C}_i$ embodies the coupling between the subsystem i and all the other subsystems. In order to make more visible the coupling of the subsystem i with every particular other subsystem, the coupling term of Equation (11) is rewritten as shown in Equation (12). The matrix $\mathbf{C}_i$ is essentially subdivided in a series of smaller sub-matrices $\mathbf{C}_{ij}$, each of them expressing the coupling between subsystem i and j.

$$\mathbf{x}_{-i}^T\,\mathbf{C}_i\,\mathbf{x}_i = \sum_{\substack{j=1\\ j\neq i}}^{m} \mathbf{x}_j^T\,\mathbf{C}_{ij}\,\mathbf{x}_i \tag{12}$$

where $\mathbf{C}_{ij}$ are a set of smaller matrices embodying the coupled terms of the design variables of designer j into designer i's model.

With this new formulation, the Rational Reaction Sets of every designer can be found, by setting to zero the first derivative of the pseudo-objective functions, as described in Equation (5). We then find unique equations for the Rational Reaction Sets of every designer. The equation of the Rational Reaction Set of designer i is shown in Equation (13) It is valid only if $\mathbf{A}_i$ is invertible; in some situations, it might not be invertible, and those cases are discussed in [20].

$$\mathbf{x}_i = -\frac{1}{2}\,\mathbf{A}_i^{-1} \sum_{\substack{j=1\\ j\neq i}}^{m} \mathbf{C}_{ij}^T\,\mathbf{x}_j - \frac{1}{2}\,\mathbf{A}_i^{-1}\,\mathbf{D}_i^T \tag{13}$$

A set of m different equations can be written similar to Equation (13), representing the Rational Reaction Set of every designer. In order to be able to use tools from Linear System Theory, we have to make the analogy between this set of equations and the main form of discrete update equation in Linear System Theory, called the *state-space equation*, and shown in Equation (14).

$$\mathbf{x}(k+1) = \mathbf{\Phi}\,\mathbf{x}(k) + \Gamma\,\mathbf{u}(k) \tag{14}$$

where $\mathbf{x}$ is the state vector (vector of variables that we are studying) and $\mathbf{u}$ the input vector. In this paper, since we are not influencing the design process in any way and just studying its dynamics, we set the input vector equal to the unity vector (corresponding to no special outside influence). The matrix $\mathbf{\Phi}$, the *state matrix* represents the dynamics of the system, how it updates from one iteration to the next. The matrix Γ, the *input matrix* embodies the influence of outside intervention, or, in our case, of initial conditions.

First, we need to write Equation (13) as a discrete-time update equation; this represents the sequential approach to the design process and is shown in Equation (15).

$$\mathbf{x}_i(k+1) = -\frac{1}{2}\mathbf{A}_i^{-1}\sum_{\substack{j=1\\j\neq i}}^{m}\mathbf{C}_{ij}^T\mathbf{x}_j(k) - \frac{1}{2}\mathbf{A}_i^{-1}\mathbf{D}_i^T \qquad (15)$$

We can now identify the set of m equations similar to Equation (15) with Equation (14). To do so, the coefficient of $\mathbf{x}_j(k)$ with the summation is identified with the matrix Φ, while the constant term is identified with Γ. Equations (16) and (17) show the expressions for the matrices Φ and Γ. Φ can be written as the multiplication of two block matrices, block i being of size n_i.

$$\Phi = -\frac{1}{2}\begin{bmatrix}\mathbf{A}_1^{-1} & & & \\ & \mathbf{A}_2^{-1} & & 0 \\ 0 & & \ddots & \\ & & & \mathbf{A}_m^{-1}\end{bmatrix}\cdot\begin{bmatrix}0 & \mathbf{C}_{12}^T & \cdots & \mathbf{C}_{1m}^T \\ \mathbf{C}_{21}^T & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathbf{C}_{m1}^T & \cdots & \cdots & 0\end{bmatrix}$$

$$= -\frac{1}{2}\,\mathrm{diag}\left(\mathbf{A}_i^{-1}\right) * \Lambda \qquad (16)$$

$$\text{where } \Lambda = \begin{cases}\lambda_{ij} = \mathbf{C}_{ji}^T \\ \lambda_{ii} = 0\end{cases}$$

$$\Gamma = -\frac{1}{2}\begin{bmatrix}\mathbf{A}_1^{-1}\mathbf{D}_1^T \\ \vdots \\ \mathbf{A}_m^{-1}\mathbf{D}_m^T\end{bmatrix} \qquad (17)$$

The formulations of these matrices look fairly complicated, but, in fact, they are straight forward, as they are only functions of the matrices involved in the objective functions of the designers. From Equations (16) and (17), Φ is a square matrix composed of blocks and its size is the sum of the size of every block which are of size n_i, which is N by Equation (4). Thus, Φ is of size $N \times N$; similarly, Γ is of size $N \times 1$.

Therefore, we now have the formulation for the two matrices Φ and Γ and a new update equation for the state vector $\mathbf{x}$ shown in Equation (18).

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma \qquad (18)$$

Once Equation (18) has been derived, it is possible to find the steady-state and the stability of the problem. The steady-state solution corresponds to the equilibrium point of the physical system studied, the design space in our case. If it exists, Linear System Theory ensures its uniqueness, given by Equation (19).

$$\mathbf{x}^* = [\mathbf{I}_N - \Phi]^{-1}\,\Gamma \qquad (19)$$

where $\mathbf{I}_N$ is the identity matrix of size N.

However, it is even more important to study the stability of this equilibrium. According to Linear System Theory, the definition of *asymptotic stability* is used [36]:

Theorem: *The equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k)$ is asymptotically stable if and only if all eigenvalues of $\mathbf{A}$ have magnitudes less than 1.*

Therefore, the stability of the equilibrium point of the design space can be expressed as a function of the spectral radius of the state matrix Φ (which is defined in Equation (10)). The convergence analysis of the design process can therefore be captured as follows: the design process converges to the equilibrium point found thanks to Equation (19) if and only if:

$$r_\sigma(\Phi) < 1 \qquad (20)$$

This result is very important, as it means that building the state matrix Φ and calculating its spectral radius gives insightful information on the convergence of the design process to the equilibrium point, which can also be calculated using Equation (19).

The next section investigates an hypothetical simple decentralized design case study to show how the methods presented in this section are applied to a decentralized design problem.

EXAMPLE PROBLEM

The case study presented here investigates a design example involving five designers with quadratic objective functions, each of them controlling a different number of design variables. This case study has been hypothetically created using Matlab to model a quadratic distributed optimization problem with relatively high interactions between the models of the designers. Tables 2-6 present the model of each designer: the design variables controlled, as well as the objective function to be minimized (which is quadratic, and can be a utility function or a response surface approximation of the actual objective function of the designer including the constraints of the subsystem).

Several interesting properties can be found by studying these models, including the coupling between the systems. For example, subsystem 5 is coupled with subsystems 2 and 4 (through the terms with x_3, x_4, x_5, x_8, and x_{10}), but completely independent of the models of designers 1 and 3. Another property worth mentioning is that *every* subsystem is dependent on subsystem 2, while subsystem 2 itself is only weakly coupled to the other subsystems. This illustrates a design process where one of the disciplines has a strong influence on the design process since it is influencing every other subsystem.

Before going further and numerically studying this design problem, it is necessary to describe some properties related to the mechanics of the design process. An important instant is that

DESIGNER 1

Design variables: x_1 and x_2

$$\begin{aligned}F_1 = {} & 9.41x_1^2+1.80x_2^2+6.06x_1+1.62x_2+8.16x_1x_3+2.82x_2x_4\\ & +6.20x_2x_5+9.82x_1x_6+4.26x_2x_7+2.37x_1x_8+1.25x_1x_9\\ & +2.23x_2x_{10}+3.47x_1x_{11}+0.23x_2x_{12}\end{aligned}$$

Table 2. DESIGNER 1 MODEL

DESIGNER 2

Design variables: x_3, x_4 and x_5

$$\begin{aligned}F_2 = {} & 6.55x_3^2+7.57x_4^2+5.68x_5^2+4.29x_3+8.84x_4+5.25x_1x_5\\ & +2.13x_5x_8+5.34x_5x_9+3.48x_5x_{13}+2.34x_3x_{14}+4.57x_4x_{15}\\ & +4.12x_5x_{16}\end{aligned}$$

Table 3. DESIGNER 2 MODEL

DESIGNER 3

Design variables: x_6 and x_7

$$\begin{aligned}F_3 = {} & 7.81x_6^2+5.49x_7^2+5.43x_6+7.51x_7+7.87x_1x_6+4.57x_2x_7\\ & +4.52x_3x_6+1.23x_4x_6+2.47x_4x_7+2.12x_5x_7+3.26x_6x_{10}\end{aligned}$$

Table 4. DESIGNER 3 MODEL

DESIGNER 4

Design variables: x_8, x_9, x_{10}, x_{11} and x_{12}

$$\begin{aligned}F_4 = {} & 9.88x_8^2+9.86x_9^2+6.49x_{10}^2+9.48x_{11}^2+6.4x_{12}^2+5.43x_8\\ & +1.23x_9+7.84x_{10}+0.32x_{11}+5.43x_{12}+1.30x_3x_9\\ & +1.94x_3x_{11}+5.75x_3x_{12}+4.32x_4x_8+0.12x_4x_{10}+4.56x_4x_{11}\\ & +3.26x_4x_{12}+4.89x_5x_8+2.3x_5x_9+1.51x_5x_{10}+3.20x_5x_{11}\end{aligned}$$

Table 5. DESIGNER 4 MODEL

DESIGNER 5

Design variables: x_{13}, x_{14}, x_{15} and x_{16}

$$\begin{aligned}F_5 = {} & 9.52x_{13}^2+7.75x_{14}^2+9.93x_{15}^2+8.12x_{16}^2+2.79x_{13}+0.5x_{14}\\ & +9.37x_{15}+6.53x_{16}+5.43x_3x_{13}+6.41x_3x_{14}+0.12x_3x_{16}\\ & +6.27x_4x_{13}+5.43x_4x_{15}+1.23x_5x_{14}+6.77x_5x_{16}\\ & +1.38x_8x_{13}+4.31x_{10}x_{14}+2.64x_{10}x_{15}+3.41x_{10}x_{16}\end{aligned}$$

Table 6. DESIGNER 5 MODEL

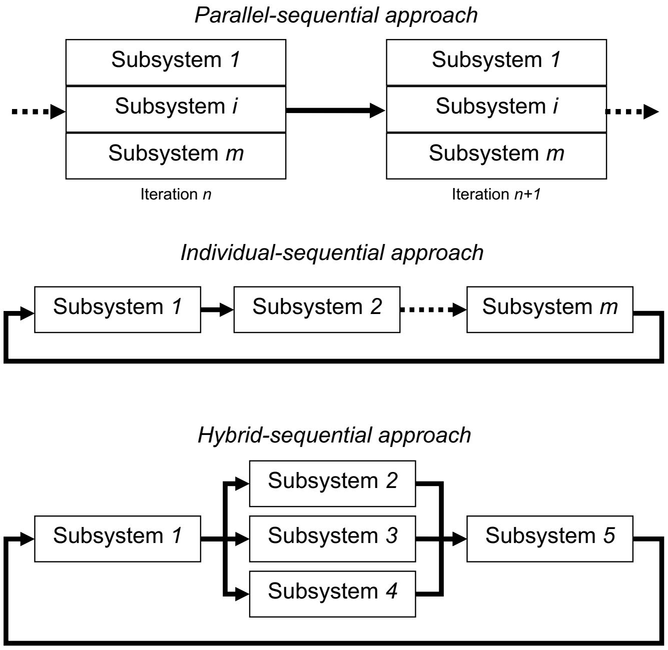

Figure 2. SEQUENTIAL APPROACHES TO DESIGN

the conditions proven in the previous sections are applicable no matter what design approach is taken. Whether it is parallel-sequential (at each time step, every subsystem solves its own model using the design variables' values obtained at the previous time step), or individual-sequential (one discipline goes after another, in a specified order), the same conditions apply to the stability of the design process. This is important because, in a design problem, the approach used can be of different sequential nature, and can also be a combination of the different types described above. The conditions developed in this paper apply in all these cases, parallel, sequential and hybrid, which are illustrated in Figure 2. Indeed, for every approach, the designers are only exchanging the design variables values between each iteration, and this is how the problem was modeled in our formulation. Even though the path taken along the design process might be different from one approach to another, the designers will still go back and forth between their RRS. Therefore the final solution will be the same for every approach and the convergence criterion developed also applies to each approach.

In order to study the stability of this design process, we first need to put the equations involved in the models of the designers in the specific form described in Equation (18). Specifically, the matrices Φ and Γ need to be written out. They are displayed in the Appendix of this paper for sake of completeness. These matrices are of size 16×16 and 16×1 respectively, since the total number of design variables in this example is sixteen.

The first step is to calculate the equilibrium point of this design problem. This is given by calculating Equation (19). Equation (21) shows the results for each of the designers.

$$\begin{aligned}
\mathbf{x}_1^* &= [-0.1499 \quad 1.3360]^T \\
\mathbf{x}_2^* &= [-0.3776 \quad -0.5084 \quad 0.2285]^T \\
\mathbf{x}_3^* &= [0.0081 \quad -1.1709]^T \\
\mathbf{x}_4^* &= [-0.2204 \quad -0.0643 \quad -0.6263 \quad 0.1055 \quad -0.1251]^T \\
\mathbf{x}_5^* &= [0.1447 \quad 0.2796 \quad -0.2494 \quad -0.3633]^T
\end{aligned} \tag{21}$$

This design point $\mathbf{x}^*$ represents the potential final point where designers might end up if they use any sequential approach to design. The stability now depends on the value of the eigenvalues of Φ. Equation (22) shows the sixteen eigenvalues of matrix Φ (there are less than sixteen because the eigenvalue $'0'$ has a multiplicity greater than 1). The spectral radius of the state matrix Φ can then be calculated using Equation (10) and compared to the convergence criterion in Equation (20).

$$\begin{aligned}
Eig(\Phi) = \{&\pm 0.7023, -0.6829, 0.48 \pm 0.1i, -0.29 \pm 0.04i, \\
&-0.3, 0.33, 0.2763, -0.0378, 0.0279, 0\}
\end{aligned} \tag{22}$$

$$r_\sigma(\Phi) = 0.7023 \tag{23}$$

Since the spectral radius of the state matrix is less than 1, it can be concluded that using any sequential design approach, and starting with any design point, the distributed design problem studied will converge to the equilibrium point given in Equation (21). This is the Nash equilibrium for this problem, and it is both individually and collectively stable, meaning that it will be the final design point that the designers will agree on at the end of the design process.

These results are verified by solving the same problem using a parallel-sequential approach, similar to the first approach shown in Figure 2. Starting the optimization with a great number of random initial conditions (different initial vectors of design variables), the distributed design process converged in all cases, thus verifying the stability found using the convergence criterion developed in this paper. The computer simulation converged if, at one iteration, the design variables values were changing by less than 0.1. With that accuracy, the simulations took, on average (depending on the initial conditions) 10 iterations to converge, meaning that each subsystem would have to solve its optimization problem ten times. Equation (24) shows the final solution for a particular run of the simulation.

$$\begin{aligned}
\mathbf{x}_1^* &= [-0.1499 \quad 1.3292]^T \\
\mathbf{x}_2^* &= [-0.3776 \quad -0.5084 \quad 0.2285]^T \\
\mathbf{x}_3^* &= [0.0082 \quad -1.1681]^T \\
\mathbf{x}_4^* &= [-0.2204 \quad -0.0643 \quad -0.6263 \quad 0.1055 \quad -0.1251]^T \\
\mathbf{x}_5^* &= [0.1447 \quad 0.2796 \quad -0.2494 \quad -0.3634]^T
\end{aligned} \tag{24}$$

Comparing it with Equation (21), it is clear that the results found with our technique match the results found using a sequential approach, as expected. We can see that some of the design variables fully converged, while some are still not converged, but are in the neighborhood of their final solution. This point is therefore the Nash solution, as presented earlier, and verifies the accuracy of the approach presented in this paper.

CONCLUSION

This paper presents a full qualitative analysis of the stability of the design process for a distributed design problem. Using concepts from Linear System Theory, conditions for stability as well as values as of the final equilibrium are found for any quadratic distributed design problem.

This paper is a strong basis for further research in this area. Future work involves finding the same kind of conditions for highly nonlinear problems, and the results are essential since they will be used as a base and reference for future research and publication. It is also worth noticing the effort for trying to bridge the gaps between different disciplines, namely, in this paper, engineering design and system theory.

We believe that this research will first need to extend the results of this qualitative analysis to more complex problems. Then, as mentioned earlier in the paper, we want to build new design techniques based on these results to help designers be more efficient and get closer to Pareto optimal final solutions even in distributed design environments.

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APPENDIX

$$\Phi = \begin{bmatrix}
0 & 0 & -0.43 & 0 & 0 & -0.52 & 0 & -0.13 & -0.07 & 0 & -0.18 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.78 & -1.73 & 0 & -1.18 & 0 & 0 & -0.62 & 0 & -0.06 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.18 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.30 & 0 \\
-0.46 & 0 & 0 & 0 & 0 & 0 & 0 & -0.19 & -0.47 & 0 & 0 & 0 & -0.31 & 0 & 0 & -0.36 \\
-0.50 & 0 & -0.29 & -0.08 & 0 & 0 & 0 & 0 & 0 & -0.21 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.42 & 0 & -0.23 & -0.19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.22 & -0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.07 & 0 & -0.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.01 & -0.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.10 & -0.24 & -0.17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.45 & -0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.29 & -0.33 & 0 & 0 & 0 & -0.07 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.41 & 0 & -0.08 & 0 & 0 & 0 & 0 & -0.28 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.27 & 0 & 0 & 0 & 0 & 0 & -0.13 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.01 & 0 & -0.42 & 0 & 0 & 0 & 0 & -0.21 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\Gamma = \begin{bmatrix} -0.32 & -0.45 & -0.33 & -0.58 & 0.00 & -0.35 & -0.68 & -0.28 & -0.06 & -0.60 & -0.02 & -0.42 & -0.15 & -0.03 & -0.47 & -0.40 \end{bmatrix}^T$$

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