Intuitive Visualization of Pareto Frontier for Multi-Objective Optimization in n-Dimensional Performance Space

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Abstract

A visualization methodology is presented in which a Pareto Frontier can be visualized in an intuitive and straightforward manner for an n-dimensional performance space. Based on this visualization, it is possible to quickly identify 'good' regions of the performance and optimal design spaces for a multi-objective optimization application, regardless of space complexity. Visualizing Pareto solutions for more than three objectives has long been a significant challenge to the multi-objective optimization community. The Hyper-space Diagonal Counting (HSDC) method described here enables the lossless visualization to be implemented. The proposed method requires no dimension fixing. In this paper, we demonstrate the usefulness of visualizing n-f space (i.e. for more than three objective functions in a multiobjective optimization problem). The visualization is shown to aid in the final decision of what potential optimal design point should be chosen amongst all possible Pareto solutions.

I. Introduction

Almost all real world engineering design problems are characterized by the presence of several conflicting and/or cooperating objectives, as opposed to having a single objective. The term Multi-objective Optimization Problems (MOP) is used to broadly classify problems with more than one objective.

A general multi-objective optimization problem can be expressed by the equations to the right where, k is the number of objective functions, n is the number of design variables, l is the number of constraints, and $S \in R^n$ is the solution or design space.

$$\min F(x) = \left(f_1(x), f_2(x), \ldots f_k(x)\right)^T$$
$$where \quad x \in S \qquad (1)$$
$$x = \left(x_1, x_2, x_3, \ldots x_n\right)^T$$
$$s.t. \, G(x) = \left(g_1(x), g_2(x), \ldots g_l(x)\right)^T$$

Typically, as a solution strategy, multi-objective problems are aggregated to form a single objective function leading to a single solution for the aggregated function. The solution found using this approach is strongly dependent on the way the objectives have been aggregated. A rather practical approach to deal with multi-objective problems is to find a set of solutions, (called a *Pareto set* shown as the *Pareto optimal frontier* in Figure 1), instead of finding a single aggregated objective-dependent global optimum. Vilfredo Pareto first introduced the concept of Pareto optima[1] in the 19th century. A solution is said to be *Pareto-optimal* if it is not dominated by any other feasible solution (i.e. there exists no other solution that is better for at least one objective function value, and equal or superior with respect to the other objective functions values). Thus, all of the Pareto solutions are equally important and all are the global optimal solutions.
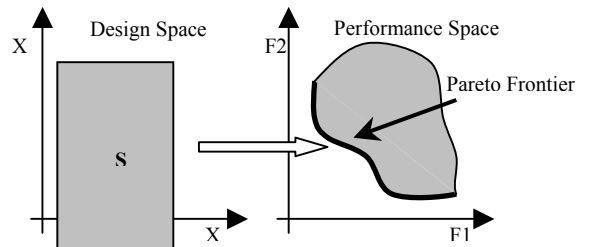
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Figure 1. Mapping from design space to performance space

The concept of Pareto optimality has been widely used in industry to aid designers in their decision-making processes. The decision-maker articulates his preference pertaining to the different objectives once he has knowledge of the Pareto frontier. The approach of visualizing the Pareto frontier has been widely used in decision-making for two and three objective problems. The Pareto frontier for two or three objective functions can be readily visualized using traditional 2-D and 3-D graphical means.

However, when the problem size is large (i.e. for more than three objective problems), there is no easy or intuitive method to visually represent the Pareto frontier. In this paper, we present a new approach that enables intuitive visualization of the Pareto frontier for n-objective problems. The method uses a new technique of 'lossless' dimension blending, termed the Hyper-Space Diagonal Counting (HSDC) method for multidimensional visualization.

II. Background

First, a brief discussion of multidimensional visualization methods is presented, followed by a discussion of the types of these methods presently used in visualizing solutions for multi-objective optimization. These discussions will provide an appropriate groundwork for understanding the Hyper-Space Diagonal Counting (HSDC) method presented here, together with its application in visualization of the n-f Pareto Frontier.

A. Multidimensional Visualization Techniques

Visualization is an incredibly powerful mechanism for providing insight into complex phenomena. By incorporating the ability to interact in real-time with resulting displays, a scientist can achieve a rapid and meaningful exploration of a problem space or dataset. The goal of Multidimensional Multivariate Visualization (MDMV) is to meaningfully translate large amounts of multidimensional data into intuitive visual representations[3]. Numerous methods and applications have been developed to achieve this goal, subject to hardware and software limitations of 2-D or 3-D visualization space. All multidimensional visualization techniques attempt to transform a multidimensional problem or dataset so that it can be mapped to a 2-D or 3-D visual space.

A standard approach that is used in multidimensional visualization is to first process the multidimensional data in such a way so as to reduce the dimensionality, while still preserving the integrity of its meaning. This is generally a particularly important step for visualization, as the associated complexity of the problem or dataset will be significantly reduced. One of the most comprehensive resources for discussion of dimension reduction techniques can be found in Carreira-Perpinan[2]. Most multidimensional visualization techniques that depend on dimension reduction in some way result in a loss of meaning, a loss of the concept of a neighborhood, and an associated loss of an ability to understand the representation in an intuitive way.

Categorizing MDMV techniques is difficult for a variety of reasons. Possible criteria for such a categorization include the goal of visualization, the type and/or dimensionality of the data, the use of color, the use of animation, and the dimensionality of the visualization technique, amongst many other possibilities. We will use the three broad categories defined in reference [3] to provide an overview of the best-known methods. These categories are summarized as follows.

1. *Techniques based on 2-variate displays as well as multiple views of 2-variate displays.* This includes fitted curves, reference grids, and banking.
2. *Techniques based on multivariate displays.* Here, color, symbols, and matrices of views are often used as a key to the representation. One of the best-known methods is the scatter plot matrix, where n dimensions are projected onto n*(n-1) scatter plots, in which each pair of dimensions has two scatterplots showing their relation[3]. HyperSlice[4] and HyperBox[5] can be considered extensions of scatterplot matrices, in which color and interactivity provide for greater insight into the problem. Another is Chernoff Faces[6] and, more broadly, the use of glyphs[7,8], to represent characteristics of relationships and the space in question. Hierarchical axis[9-11], Dimension Stacking[12], and World within Worlds[13, 14] all use the concept of a hierarchical representation in some way for the dimensional relationships. Parallel Coordinates[15-17] has become an extensively used approach in a variety of fields and applications, including for Multi-objective optimization. In this approach, a point in N-dimensional space is equivalent to a polyline through N parallel coordinates. Stump and Simpson[18] use glyphs and colors to represent multivariate data. These represent only a small number of the methods that have been proposed for n-dimensional visualization, but are probably the most widely recognized.
3. *Techniques using time as an animation parameter.* In this class of methods, time is generally used as a fourth dimension and each set of data is applied to each step of a time series which is then represented as an animation sequence.

Each of the techniques developed under the three broad categories listed above have advantages and limitations. Some of them are somewhat difficult to understand, some of them are computationally expensive, and some are just awkward. A few of the more widely used techniques are shown in Figure2. We will attempt to demonstrate the great potentials of visualization based on our HSDC method in this paper.

This paper focuses on the application of HSDC method to multi-objective optimization, in order to demonstrate its usefulness in visualizing n-f space (i.e. for more than three objective functions in a multiobjective optimization problem). In the next section, some of the leading approaches for visualizing solutions for MOPs are presented.
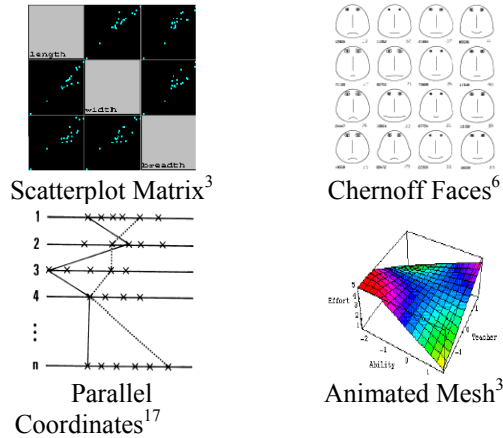
Scatterplot Matrix[3]

Chernoff Faces[6]

Parallel Coordinates[17]

Animated Mesh[3]

Figure 2. Representative visualizations associated with key MDMV visualizations.

B. Multidimensional Visualization in Multiobjective Optimization

Multi-Objective Optimization Problems (MOPs), in contrast to single objective problems, involve a set of objectives that might be cooperative, competitive or have no relationship. Typically, the case of competitive objectives is the most interesting, since the choice of an 'acceptable' or 'best' solution depends on the preferences on, compromises between, and trade-offs of the objective functions. When the objectives in an MOP are conflicting, the set of optimal solutions is known as the Pareto set, wherein no one solution is superior to the others.

The solution validation for multi-objective problems has continued to be an extremely difficult issue for all researchers.   So far, the most practical way to validate MOP solutions are to use visualization to verify the obtained results. Also, visualization allows designers to interpret results and use design information in more meaningful ways. Since visualization is recognized as a process that transforms data into visible objects, decision-makers can easily view and analyze a rich set of information and then use this to help make trade-offs so as to reduce the time and cost of the design process[18]. Multidimensional data visualization is a promising research area for application to MOPs.

Being able to visualize the Pareto frontier in a performance space with more than three objective functions has been a great challenge to the optimization community. Parallel coordinates[15-17] has been one of the leading approaches used in industry to assist in ultimately choose leading design candidates to explore further. Multivariate data is displayed using a polyline that intersects equally spaced vertical axes. Each polyline represents one design possibility. The use of parallel coordinates to visualize the performance space still remains unwieldy for large numbers of functions. Cloud Visualization[19] provides a means by which a designer can view all previously generated design information in both the design and the performance spaces simultaneously. However, since all spaces are displayed in separate windows that are linked, the functionality is compromised for large numbers of functions, since it becomes tedious to work with too many windows. Mattson and Messac[20] use a Pareto filter to reduce the Pareto frontiers from various disparate design concepts into a single Pareto frontier termed the *s-Pareto frontier*. Unfortunately, the use of approximations in the s-Pareto approach ultimately results in loss of dimension representation.

All these visualization techniques, which are essentially used as methods of solution validation, become cumbersome for very large problems. Hence, developing an appropriate multidimensional visualization capability to facilitate the decision-making process for large multi-objective optimization problems is the main focus of this work.

III. Development of a Framework for Intuitive Multidimensional Visualization (HSDC Method)

In the previous sections, a background on multi-objective optimization, multidimensional visualization techniques, dimension reduction techniques, and the application of multidimensional visualization to MOP was presented to lay the groundwork for this research. Here, the Hyper-Space Diagonal Counting (HSDC) method is developed, which enables a lossless mapping of dimensions using a counting strategy derived from complexity theory. Hence, complexity theory is briefly reviewed together with Cantor's original counting method, which is critical for understanding the HSDC method developed in this work. The HSDC method ultimately enables an intuitive and meaningful visualization capability for multidimensional multi-objective optimization problems.

The goal of complexity theory is to understand what makes some problems algorithmically difficult to solve. Research pertaining to counting complexity classes has been demonstrated to be very promising. Such research on counting complexity classes has provided important insights into the inherent complexity of many natural computational problems. These classes essentially consist of functions quantifying the time or space required to enumerate them. **Why is this important to know here?** Because the concept of counting complexity classes originated with the famous German mathematician, Georg Ferdinand Ludwig Philipp Cantor, who recognized that every point on a surface has a corresponding point on a line, and vice versa. The result of this observation is that there is a one to one correspondence of points on the interval [0, 1] and points in an n-dimensional space. It is this correspondence which provides the basis for the development of the Hyper-Space Diagonal Counting method.

Cantor's discoveries in set theory rest upon a very simple idea – that even though you may not be able to count something, you can equate it with something else that has the same cardinality (i.e. the same number of elements). Cantor's insight was that even though an infinite set can't be enumerated its cardinality can be determined or at least equated to that of another set. Two infinite sets can be shown to have the same cardinality by finding a one-to-one mapping between the elements of each set. Cantor's theorem for mapping points in a 2-dimensional space on a line can be written as in Eq. 2. The cardinality of $|N|$ denotes $\aleph_0$ (the meta-letter aleph-zero or aleph-null), which is equal to the cardinality of the integers.

$$|N| = |N^2| \tag{2}$$

where, $N = \{1, 2, 3, 4, ...\}$
$N^2 = \{ordered\ pairs\ of\ members\ of\ N\}$

The very famous proof is represented visually in Figure 3. Consider an array that includes all the ordered pairs of positive integers and then make a path through all the pairs in the spiral way shown. In this way, every single element of set N2 can be mapped to one (and only one) element of set N (i.e. the cardinality of set N is same as that of set N2). Hence, for every point on a 2-dimensional surface there is a corresponding point on a line, which is unique. This concept of counting, which represents a very small portion of Cantor's contributions to the field of mathematics, is key to understanding the Hyper-Space Diagonal Counting (HSDC) Method we propose here. The use of the HSDC enables us to achieve lossless dimension blending for multidimensional visualization.
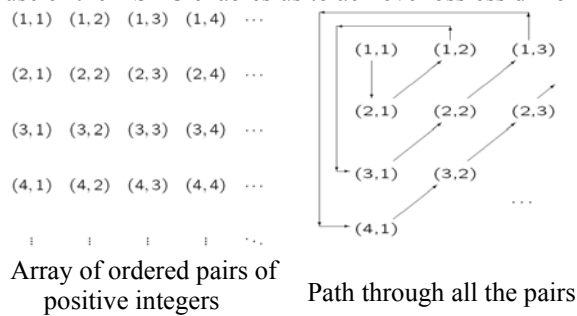
(1,1)	(1,2)	(1,3)	(1,4)	⋯
(2,1)	(2,2)	(2,3)	(2,4)	⋯
(3,1)	(3,2)	(3,3)	(3,4)	⋯
(4,1)	(4,2)	(4,3)	(4,4)	⋯

Array of ordered pairs of positive integers   Path through all the pairs
Figure 3. Graphic proof of Cantor's theorem

We have extended Cantor's work on counting to enable mapping points in an n-dimensional space to a line. Incredibly, this has not been done, as others who use counting in Complexity Theory have had no need to move beyond two dimensions. This mapping can be described by:

$$|N| = |N^2| = |N^3| = |N^4| \ldots = |N^n| = \aleph_0 \tag{3}$$

where,
$N = \{1, 2, 3, 4, ...\}$
$N^n = \{ n^{th}\ ordered\ combinations\ of\ all\ the\ members\ of\ N \}$

The proof for this is almost unnecessary as Cantor has already done it for N^2 and what remains is a matter of making an array of nth ordered combinations of all the members of N^n and then creating a path through all the points to get a sequence. An extension of Figure 3 to show a mapping for three-dimensional points is shown in Figure 4. In order to understand how this counting will be generalized, we have defined the following terms that will be used throughout our method development section. The variable n is used to denote the number of dimensions, l is used to represent the level on which a particular point falls (i.e. which loop out from the origin), E_l is the number of elements (i.e. points) on a particular level, C_i is the counting index associated with element i (point i), and TE^n_l are the total number of elements up to level l.
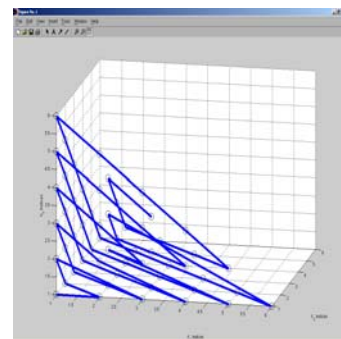
Figure 4. Path for counting in 3-d space

There are five equations which must be developed in order to be able to automate a counting in any multidimensional space. We need to know the following:

1) the total number of elements at a particular level;
2) the total elements up to a particular level;
3) the elements at a particular level in terms of the elements of the previous level;
4) the total number of elements up to a level in terms of the elements in that level; and
5) what counting indices will correspond to a particular level.

The first four equations and the fifth formula have been developed as is shown in Equations 4-8.

$$E_l^{\#} = \begin{cases} \left(\displaystyle\prod_{k=0}^{k=n-2}(l+k)\right) \Big/ (n-1)! & n \neq 1 \\[4mm] 1 \Big/ (n-1)! & n = 1 \end{cases} \quad (4)$$

$$TE_l^n = \sum_1^l E_l^n \quad (5)$$

$$E_{l+1}^n = \left(\frac{l+n-1}{l}\right)E_l^n \quad (6)$$

$$TE_l^n = \left(\frac{n+l-1}{n}\right)E_l^n \quad (7)$$

The final formula provides more of a challenge. It can be expressed in terms of an inequality as:

$$C_i \leq \frac{(l+n-1)!}{(l-1)!\,(n)!} \quad (8)$$

The fifth formula (Eq. 8) is an (n-1)th order polynomial inequality, which must be solved in constant time in order to automatically generate any combination of variables for any size multidimensional problem. Alternatively, an optimization routine or similar technique could be used in a preprocessing mode. This would not be as desirable an approach, although it would certainly work.

What this counting and indexing approach enables us to do is to map each and every dimension in a 'lossless' way (i.e. no dimension fixing) to a 2-D or 3-D graphical display. Because the counting is done in an outward spiraling manner, the concept of a neighborhood is preserved with the HSDC, as is the concept of going from small to large as one moves from left to right in the indexing. The details of how the HSDC method is applied to MOP is presented in the next section.

IV.   HSDC Applied to Visualization of Multidimensional Performance Space in Multiobjective Optimization

We first apply the HSDC approach to a two function multi-objective optimization problem to demonstrate that the resulting visualization of the Pareto frontier is what we are used to seeing for 2-D performance space. We will then extend the method to two different three objective problems and then a four objective problem.

A. Two Objective Test Problem

Consider a test problem with two objectives, four design variables, and two inequality constraints, in which the objectives are competitive and to be minimized. The problem statement is shown in Eq. 9. A typical 2-D visualization of the Pareto frontier for this problem is a bi-objective plot as shown in Fig. 5a or the parallel coordinate plot seen in Fig. 5b. Both these visualizations are incredibly useful, but they have their own limitations. For the visualization of Figure 5, we can plot the Pareto frontier for up to only three objectives (if we move to a 3-D representation). Using parallel coordinates, we can theoretically have as many objectives on parallel axes as needed. However, for large problems, we soon have so many of these plots that an intuitive interpretation becomes impossible.

Minimize :

$$f_1 = 2 + (X_1 - 7)^2 + \left(\frac{X_2}{2} - 6\right)^2 - 3X_3 + (X_4 - 5)^2$$

$$f_2 = -10X_1 - X_2 + 2(X_3 - 7)^2 - 5X_4 \quad (9)$$

Subjected   to :

$$g_1 = \frac{\left((X_1 - 5)^2 + (X_3 - 2)^2 - 75\right)}{75} \leq 0$$

$$g_2 = \frac{(2X_1 + 2X_2 + 2X_3 + X_4 - 70)}{70} \leq 0$$

Where : $0 \leq X_i \leq 20$

As we established in the previous sections, the HSDC method can easily map *n* dimensions to a line, without loss of dimensional representation. In its application to multi-objective optimization, we can readily use the counting scheme described to map any number of objectives to a single line in order to generate the visual representation of the Pareto frontier for any number of objectives. The procedure developed to enable visualization of the Pareto frontier for multidimensional performance space is outlined next. The counting scheme is implemented together with a simple but very significant binning technique in order to enable this meaningful visualization.

For demonstration purposes, consider the same bi-objective problem described previously. The steps for visualizing the Pareto frontier using HSDC are outlined.

Step 1. Obtain Pareto points using any appropriate optimization routine.

Step 2. Identify the minimum and maximum values for each of the objectives to establish a range. Divide these ranges into some finite number of compartments, resulting in small bins along each objective.(e.g. F1 range divided into five, resulting in five bins in which a point may fall).

Step 3. The indices of the bins created can be plotted on a line and thus we can have F1 bin indices on one axis and F2 bin indices on the other axis (e.g. for five ranges associated with F1, there will be five bins and, hence, five indices – 1 through 5). For problems with more than two objective functions, multiple functions are represented on a single axis through the use of counting in the HSDC method.

Step 4. Each Pareto point previously generated in Step 1 will fall under some combination of these bins that are plotted using their indices on the X and Y axes (e.g. if a Pareto point has an F1 that falls into the third range of F1 and an F2 that falls into the fourth range of F2, this would correspond to indices (3,4) and would be placed in that particular bin). The points are represented as a unit cylinder along the vertical axis. Multiple solutions might fall at the same set of indices, resulting in a bin that contains multiple Pareto points.
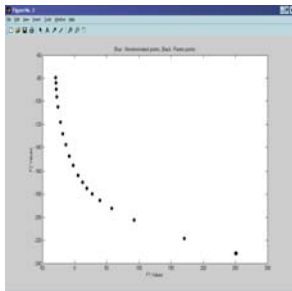
Figure 5a. Pareto frontier for the above problem using the f1, f2 performance space
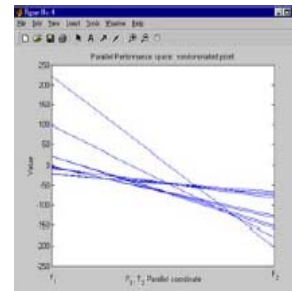
Figure 5b. Parallel coordinate plot of the performance space for the same problem

Figure 6 shows the resulting 2-D representation of the Pareto frontier for the bi-objective problem using the HSDC method. This provides a representation that is exactly what would be expected using traditional means of plotting (i.e. Figure 5a). The only difference between the two approaches is that the HSDC representation is actually plotting per index, not per functional value. Hence, due to this discretization, some small changes in the visual representation might occur. What's important to note, however, is that the axes are actually enumerating the indices. Hence, this is easily extensible to more than two objective functions. For instance, in a four objective function MOP, we can index two objectives per axis. For six objective functions, we would index three objective functions per axis. The resulting plot, however, will still look very similar to the traditional Pareto frontier shown below.

A slight variation of this approach yields an extremely powerful capability. Rather than running some number of optimizations in order to generate Pareto points, it is possible to sample points in the original design space, evaluate the objectives at these points, and then identify which of these are non-dominated. Then, the same procedure outlined in Steps 2 through 4 can be performed. While any type of approach could be used to sample the design space, the HSDC method ensures capture of all dimensions existing in the original space.
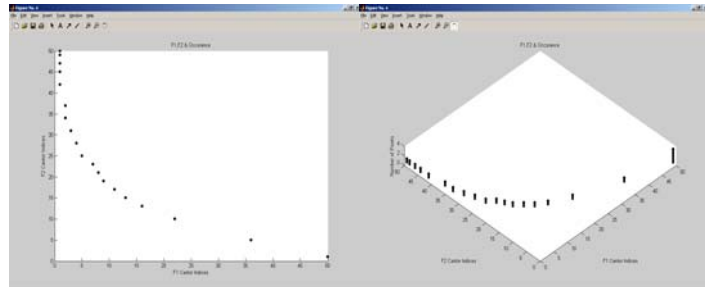

Figure 6. Pareto points (bins) for two objective performance space using HSDC method.

For instance, consider a case for the same bi-objective optimization problem we've been discussing, where a discretization grid of 10 is used along all the dimensions (design variables X_1 through X_4). Objective function values are evaluated at all the grid points. All the feasible points are compared to enumerate the non-dominated points. The blue points in Figures 7a and 7b represent the non-dominated points for the problem and the green points show all the feasible points.

Figure 7a shows a more coarse discretization than does Figure 7b, with the non-dominated blue points falling in lower indices bins. This is what we would normally expect since the non-dominated points have low objective function value for at least one of the objectives. Figure 7b shows a more refined representation, with 50 bins along

each objective. We can readily see that non-dominated points form an approximate Pareto curve by falling towards the low objective valued bins. This representation is clearly analogous to the normal Pareto frontier representation from Figures 5a and 6. The motivation for using this approximate approach would be for applications where obtaining a full set of Pareto optimum points is difficult or impractical. We can see that the approximate Pareto frontier provides valuable information to the designer, suggesting where further refinement and investigation might be warranted.
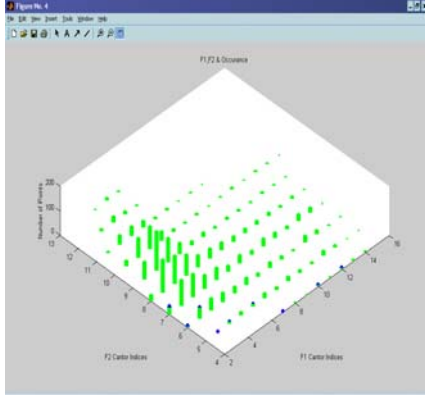
Figure 7a. F_1 bins on X-axis, F_2 bins on Y-axis and occurrence of feasible and non-dominated points on Z-axis (Grid of 20 for F_1 & F_2 bins)
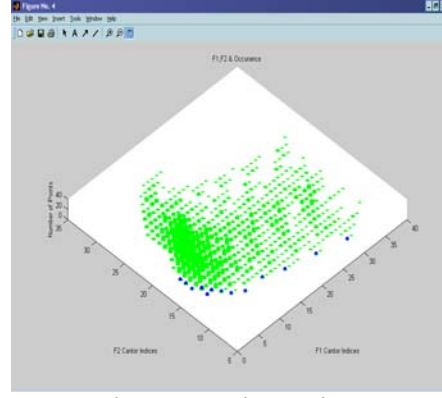
Figure 7b. F_1 bins on X-axis, F_2 bins on Y-axis and occurrence of feasible and non-dominated points on Z-axis (Grid of 50 for F_1 & F_2 bins)

Figure 7c shows the true Pareto points (in black) for the problem along side the non-dominated points (in blue) obtained using the grid. Also, as we can see from the previous figures that the larger the number of divisions we have for our bins, the better the shape for the Pareto frontier. The division of the objectives for an optimum number of bins remains a research issue. Another issue pertains to the grouping of objectives for indexing. This is discussed further in subsequent sections. Next, the HSDC approach for MOP Pareto frontier visualization is applied to a multi-objective problem with three objective functions.
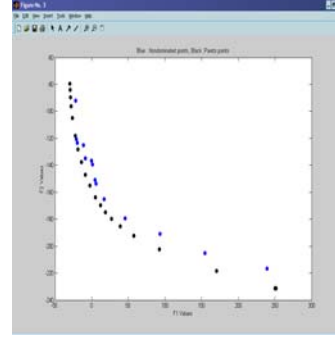

Figure 7c. F_1 on X-axis, F_2 on Y-axis; Showing Pareto points & Non-dominated points

B. Three Objective Test Problem (3 constraints)

This MOP has three objectives, three inequality constraints, side constraints, and two design variables. The problem statement is given in Eq. 10. For this problem, true Pareto points were found, together with results of an HSDC-based evaluation to provide non-dominated and feasible points (as explained in the last section). Here, F1 and F2 are indexed on one axis while F3 is indexed on the second. Figure 11 shows feasible (green) and non-dominated (blue) points generated through evaluation at index points in the discretized design space (using HSDC), as well as true Pareto points (black) obtained as a result of optimization runs.

$Minimize$:

$$f_1 = \frac{(X_1 - 2)^2}{2} + \frac{(X_2 + 1)^2}{13} + 3$$

$$f_2 = \frac{(X_1 + X_2 - 3)^2}{175} + \frac{(2X_2 - X_1)^2}{17} - 13$$

$$f_3 = \frac{(3X_1 - 2X_2 + 4)^2}{8} + \frac{(X_1 - X_2 + 1)^2}{27} + 15 \qquad (10)$$

$Subjected\ to$:

$$g_1 = 4X_1 + X_2 - 4 \leq 0$$

$$g_2 = -1 - X_1 \leq 0$$

$$g_3 = X_1 - X_2 - 2 \leq 0$$

$Where$: $-4 \leq X_1, X_2 \leq 4$

Figure 11 shows that the HSDC-based indexed representation is quite intuitive, clearly showing the Pareto frontier. Further, using the HSDC in the design space allows for the evaluation of objective functions with a subsequent identification of non-dominated points. These points clearly approximate the behavior of the Pareto

frontier itself, without having to actually perform any formal optimization. Hence, for the purposes of identifying 'good' regions for further detailed exploration, this approximate Pareto frontier approach might be worthwhile.

It should be noted that having an odd number of objectives necessitates a slightly different indexing strategy so that the resulting representation is still visually viable. This is a minor point, which can be seen from the indices in the figures below. Another point to recall is that each 'point' in Figure 8a is actually a bin, which might contain multiple points (as shown more clearly in Figure 8b). The tremendous advantage of using the HSDC is that each point within a bin can be linked back to its original design space values. Hence, a bin with multiple 'points' in the performance space allows the designer to go back to the design space values
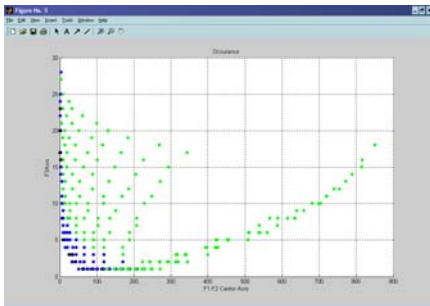
Figure 8a. F_1F_2 on X-axis, F_3 on Y-axis, number of points on Z-axis (Grid of 30 for all three objectives)
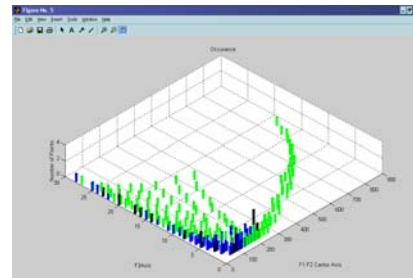
Figure 8b. F_1F_2 on X-axis, F_3 on Y-axis, number of points on Z-axis (Grid of 30 for all three objectives), both non-dominated and Pareto points.

Figures 8c-8d show Pareto points only, with a discretization of 50 for each of the three objective functions. There are 49 Pareto points for this particular problem, which are therefore being distributed amongst 50x50x50 (125,000) bins. One can see the slight differences in the Pareto frontier between Figures 8a and 8c, due to the difference in binning (27,000 versus 125,000). This is further demonstrated in Figure 8e, in which each function has 80 discretizations (for a total of 512,000 possible bins into which the Pareto points might fall). However, the Pareto frontier is still easily and intuitively recognized, even for the 30 bin discretization.
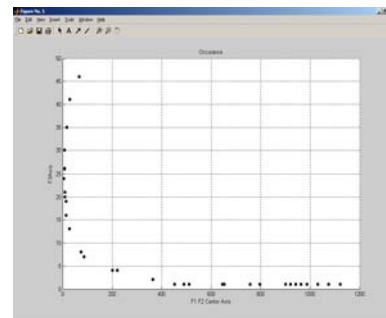

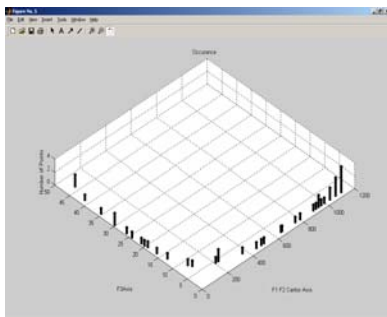**Figure 8c. F_1F_2 on X-axis, F_3 on Y-axis, number of points on Z-axis (Grid of 50 for all three objectives)**

Figure 8d. F_1F_2 on X-axis, F_3 on Y-axis, number of points on Z-axis (Grid of 50 for all three objectives)
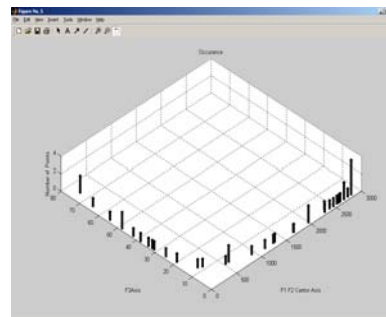

Figure 8e. F_1F_2 on X-axis, F_3 on Y-axis, number of points on Z-axis (Grid of 80 for all three objectives)

C. Three Objective Test Problem (one constraint)

Another three objective MOP was investigated with one constraint and three design variables. The problem formulation is shown in Eq. 11 below. Again, we see from Figs. 9a-c that a clear Pareto frontier is obtained, even while representing this three objective problem in a 2-D graphical display. Recall that the third dimension in the figures shown corresponds to the number of Pareto points falling into a particular bin. Also, we see from the three figures that a more refined discretization results in more bins with fewer points falling per bin.

$Minimize:$

$$f_1 = 25 - \frac{\left(X_1^3 + X_1^2\left(1 + X_2 + X_3\right) + X_2^3 + X_3^3\right)}{10}$$

$$f_2 = 35 - \frac{\left(X_1^3 + 2X_2^3 + X_2^2\left(2 + X_1 + X_3 + X_3^3\right)\right)}{10} \quad (11)$$

$$f_3 = 50 - \frac{\left(X_1^3 + X_2^3 + 3X_3^3 + X_3^2\left(3 + X_1 + X_2\right)\right)}{10}$$

$Subjected \quad to:$

$$g_1 = 12 - X_1^2 + X_2^2 + X_3^2 \leq 0$$
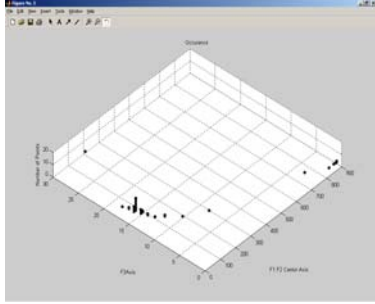
$Where: \quad 0 \leq X_1, X_2, X_3 \leq 10$

Figure 9a. F₁F₂ on X-axis, F₃ on Y-axis, number of points on Z-axis (Grid of 30 for all three objectives)
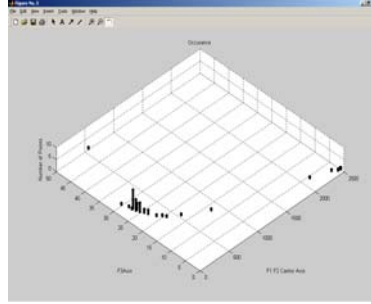

Figure 9b. F₁F₂ on X-axis, F₃ on Y-axis, number of points on Z-axis (Grid of 50 for all three objectives)
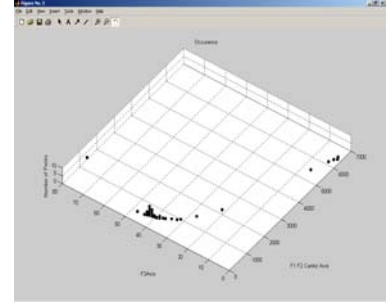

Figure 9c. F₁F₂ on X-axis, F₃ on Y-axis, number of points on Z-axis (Grid of 80 for all three objectives)

D. Four Objective Test Problem

The four objective problem in this paper derives originally from a three objective test problem taken from Veldhuizen's dissertation[21]. Another objective was added using the same two design variables. The three inequality constraints for the test problem remained the same, as do the side constraints. The problem statement is shown in Equation 11 to the right.

Figures 10a-b show two orientations of a Pareto frontier for the case where F1 and F2 are indexed along the x axis and F3 and F4 are indexed along the y axis, with a coarse 15 discretizations for each of the objective functions. Figure 10c corresponds to a finer discretization of 40 along each objective. These figures clearly show a Pareto frontier which has the exact same meaning as that for the previous bi-objective problem. As the grid is discretized to a finer resolution, we see that the greater number of available bins for points

$Minimize:$

$$f_1 = \frac{(X_1 - 2)^2}{2} + \frac{(X_2 + 1)^2}{13} + 3$$

$$f_2 = \frac{(X_1 + X_2 - 3)^2}{175} + \frac{(2X_2 - X_1)^2}{17} - 13 \quad (11)$$

$$f_3 = \frac{(3X_1 - 2X_2 + 4)^2}{8} + \frac{(X_1 - X_2 + 1)^2}{27} + 15$$

$$f_4 = \frac{(3X_1 + X_2 + 9)^2}{34} + \frac{(X_1 + 1)^2}{15} + 29$$

$Subjected \quad to:$

$$g_1 = 4X_1 + X_2 - 4 \leq 0$$

$$g_2 = -1 - X_1 \leq 0$$

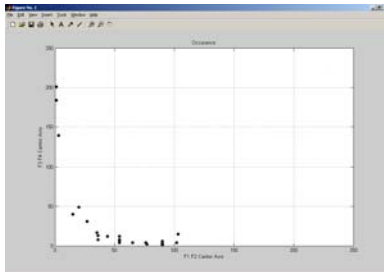$$g_3 = X_1 - X_2 - 2 \leq 0$$

$Where: \quad -4 \leq X_1, X_2 \leq 4$

to fall in results in a smoother visual representation of the Pareto frontier. This type of visualization can clearly provide a valuable means for enabling designers to determine which of the Pareto points obtained should be investigated further for an ultimate optimal solution based on designer preferences.

Figure 10a. F₁F₂ on X-axis, F₃F₄ on Y-axis, number of points on Z-axis (Grid of 15 for all four objectives)

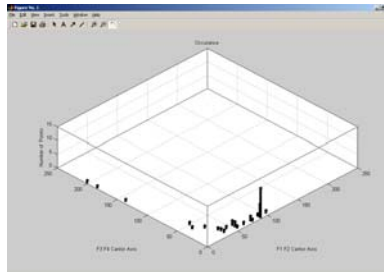**Figure 10b. F₁F₂ on X-axis, F₃F₄ on Y-axis, number of points on Z-axis (Grid of 15 for all four objectives)**

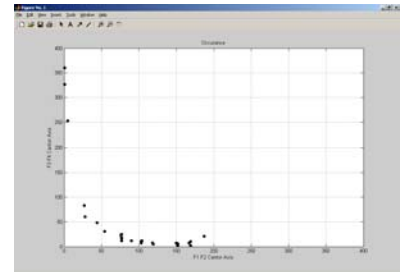**Figure 10c. F₁F₂ on X-axis, F₃F₄ on Y-axis, number of points on Z-axis (Grid of 40 for all four objectives)**

Additionally, the approach of using HSDC in the design space to evaluate functions for approximate Pareto frontier visualization is also implemented. A grid is created in the design space, just as we did for the previous problem. Subsequently the function values for all the objectives are evaluated at each of the grid points. The maximum and minimum values for each objective are calculated to determine the ranges. The ranges obtained are then divided to form the bins. It should be noted that while creating bins for the various objectives we also associate indices with increasing order of their values. The next step is to use the HSDC method on the indexes of the objective bins to collapse the four objectives on to the horizontal plane, two each on the X and Y-axes. Each index on the X and Y-axes would be associated with a combination the indices of the bins of two objectives using the Hyper-Space Diagonal Counting. In reality, we can have as many objectives on one axis as we need by creating a path through the indices of the bins created for each objective.

The design space pertaining to each objective is shown in Figure 11. The red points are the infeasible points. Feasible points are shown in green and the blue points are the non-dominated points. The performance space plot for the four objective problems using the HSDC method is shown for a discretization of 10 in Fig. 12 and for a discretization of 15 in Figure 13.

It can be readily seen from these figures that the non-dominated points form an approximate Pareto frontier along the indices of the objective bins which have low objective function values. The more the number of bins along the objectives, the better is the distribution of non-dominated points along the indexed performance space. But it should be noted that there is a trade-off in having a greater number of bins, as we would need to compare the values of the non-dominated points with more and more numbers of ranges of these bins. This results in an associated computational cost which grows incredibly fast with more objective functions and increased discretizations.
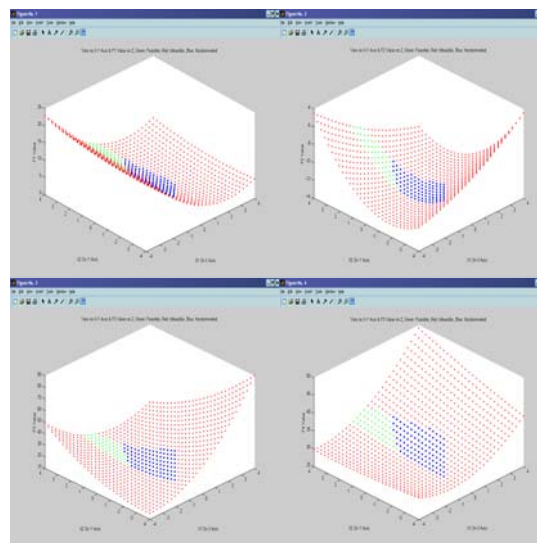

Figure 11. X_1 and X_2 on X&Y-axis, and Objective functions F1, F2, F3, and F4 on Z-axis
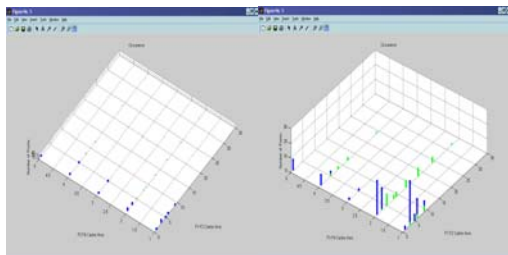

Figure 12. F_1F_2 on X-axis, F_3F_4 on Y-axis, number of points on Z-axis (Grid of 10 for all four objectives)
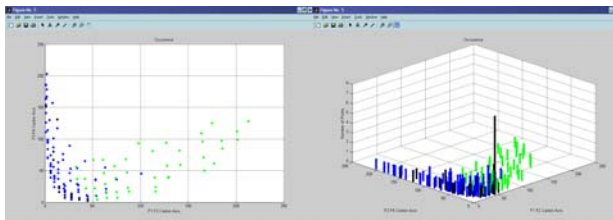

Figure 13. Two views of F_1F_2 on X-axis, F_3F_4 on Y-axis, number of points on Z-axis (Grid of 15 for all four objectives)

V. Conclusions

In this paper, we presented a powerful new approach for visualizing the Pareto frontier for problems with more than three objectives. The results presented here clearly demonstrate that the HSDC method is a feasible and usable approach for visualizing the Pareto frontier for MOPs with more than three objectives. While the question of scalability to larger MOPs is of issue, the method can theoretically visualize n-objectives meaningfully. The HSDC-based Pareto frontier visualization capability described here is particularly important for large-scale multiobjective optimization problems. The method presented overcomes the issues of awkwardness and dimension fixing so prevalent in the existing visualization strategies. The HSDC-based approach is a lossless representation, in which every single dimension is represented in a 2- or 3-D visual representation which is intuitive and easy to understand. This method represents a significant breakthrough for visualizing the Pareto frontier for multi-objective optimization methods with more than three objective functions.

VI. Acknowledgments

VII. References

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