1- A WSS random process $X(t)$ with autocorrelation $R_{XX}(\tau) = Ae^{-a|\tau|}$, where A and a are real positive constants, is applied to the input of an linear time-invariant (LTI) system with impulse response $h(t) = e^{-bt}u(t)$, where b is a real positive constant. Find the power spectral density of the output $Y(t)$ of the system.

Solution:

$H(\omega) = F[h(t)] = \frac{1}{j\omega + b} \Rightarrow |H(\omega)|^2 = \frac{1}{\omega^2 + b^2}$. Also $S_{XX}(\omega) = F[R_{XX}(\tau)] = A\frac{2a}{\omega^2 + a^2}$.

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = \left(\frac{1}{\omega^2 + b^2}\right)\left(\frac{2aA}{\omega^2 + a^2}\right) = \frac{aA}{(a^2 - b^2)b}\left(\frac{2b}{\omega^2 + b^2}\right) - \frac{A}{a^2 - b^2}\left(\frac{2a}{\omega^2 + a^2}\right).$$

2- Consider a random process given by $X(t) = A\cos(\omega t + \theta)$, where A and ω are constants, and Θ is a uniform random variable over $[-\pi, \pi]$. Show that $X(t)$ is WSS.

Solution:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & \text{for } -\pi \le x \le \pi \\ 0, & \text{otherwise} \end{cases}.$$

For WSS we need to show

1- $E[X(t)] = \overline{X}$ = constant,

2- Autocorrelation depends only on the time difference τ. $E[X(t)X(t+\tau)] = R_{XX}(\tau)$.

$E[X(t)] = \overline{\int_{-\infty}^{\infty} A\cos(\omega t + \theta) f_\Theta(\theta) d\theta} = \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta) d\theta = 0$, which satisfies the first condition.

$$R_{XX}(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\cos(\omega t+\theta)\cos(\omega(t+\tau)+\theta)d\theta = \frac{A^2}{2}\cos\omega\tau,$$

which satisfies the second condition.

Bonus: In problem 2 above, show that the random process $X(t)$ is ergodic in both the mean and the auto-correlation.

For the bonus: up to two extra points.

Solution:

IA random process $X(t)$ is ***ergodic*** if time-averages are the same for all sample functions, and are equal to the corresponding ensamble averages.

n an ergodic process, all its statistics can be obtained from a single sample function.

A stationary process $X(t)$ is called ***ergodic in the mean*** if $\langle x(t)\rangle = \overline{X}$,

and ***ergodic in the autocorrelation*** if $\langle x(t)x(t+\tau)\rangle = R_{XX}(\tau)$.

$$\langle x(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}x(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}A\cos(\omega t+\theta)dt = \lim_{T\to\infty}\frac{1}{T\omega}\int_{-\omega T/2}^{\omega T/2}A\cos(\omega t+\theta)d\omega t =$$

$\frac{A}{2\pi}\int_{-\pi}^{\pi}\cos(\omega t+\theta)d\omega t = 0$ which is equal to $E[X(t)]$.

Similarly, $\langle x(t)x(t+\tau)\rangle = \frac{A^2}{2}\cos\omega\tau = R_{XX}(\tau)$