1- Given five devices with life-time described by the density function: $f_T(t) = \alpha e^{-\alpha t} u(t)$, $\alpha > 0$. One of them is put in service at time $t = 0$, and when it fails, it is replaced by the second device, and so on until all devices are used. Find the expected value and the variance of the combined life-time of the five devices. Assume that their individual life-times are independent random variables, and that the life-time is measured from the moment the device is put in service.

Solution:

Let $T = T_1 + T_2 + \ldots + T_5$. Since independent, $f_T(t) = f_{T_1}(t_1) * f_{T_2}(t_2) * \ldots f_{T_5}(t_5)$.

$\phi_T(s) = \left(\frac{\alpha}{\alpha - s}\right)^5$. $\phi'_T(s) = \alpha^5 \cdot 5(\alpha - s)^{-6}$. $\phi''_T(s) = \alpha^5 .5.6(\alpha - s)^{-7}$.

$\phi'_T(0) = \alpha^5 \cdot 5(\alpha)^{-6} = \bar{T} = \frac{5}{\alpha}$. $\phi''_T(0) = \alpha^5 .5.6(\alpha)^{-7} = \frac{30}{\alpha^2} = \overline{T^2}$

$$\sigma_T^2 = \overline{T^2} - \bar{T}^2 = \frac{30}{\alpha^2} - \frac{25}{\alpha^2} = \frac{5}{\alpha^2}$$

2- Two independent machines are put in service at $t = 0$. Let T_1 = time of failure of #1, T_2 = time of failure of #2.

Assume $f_{T_1}(t_1) = ae^{-at_1}u(t_1)$, $a = \frac{1}{100}$, and $f_{T_2}(t_2) = be^{-bt_2}u(t_2)$, $b = \frac{1}{200}$.

What is the average length of time that at least one machine is operating?

Solution:

Define $T = max(T_1, T_2)$. $\bar{T} =$ average length of time that at least one machine is working.

$$F_T(t) = \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t} dt_1 \; f_{T_1, T_2}(t_1, t_2).$$

Since Independent $$= \int_{-\infty}^{t} dt_2 \left(\int_{-\infty}^{t} dt_1 \frac{1}{100} e^{-t_1/100} \right) \frac{1}{200} e^{-t_2/200} = (1 - e^{-t/100})(1 - e^{-t/200})u(t) \quad .$$

$$f_T(t) = \frac{d}{dt}(1 - e^{-t/100} - e^{-t/200} + e^{-3t/200})u(t) = \frac{1}{100}e^{-t/100} + \frac{1}{200}e^{-t/200} - \frac{3}{200}e^{-3t/200}u(t).$$

$$\bar{T} = \int_{0}^{\infty} t f_T(t) dt = 100 + 200 - \frac{200}{3} = \frac{700}{3} = 233.\dot{3}$$