Relationship Between Joint R.V's

Scatter Diagrams:

$f_{X,Y}(x,y)$ tells us everything about relation between X and Y. But to have a useful parameter to describe relationship concisely, we define:

Covariance: $\boxed{cov(X,Y) = E[(X-\overline{X})(Y-\overline{Y})] = \overline{XY} - \overline{X}\,\overline{Y}}$

$$cov(X,Y) = E[(X-\overline{X})(Y-\overline{Y})] = E[XY] - \overline{Y}E[X] - \overline{X}E[Y] + \overline{X}\,\overline{Y} = \overline{XY} - \overline{X}\,\overline{Y}$$

*Def: X and Y are said to be **uncorrelated** if $cov(X,Y) = 0$ or equivalently $\overline{XY} = \overline{X}\,\overline{Y}$.*

Observe:

1- Independent R.V's are always uncorrelated. If X and Y are independent, then $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

$$E[XY] = \int dx \int dy\, f_{X,Y}(x,y)xy = \int dx \int dy\, f_X(x)f_Y(y)xy = \left(\int dx\, f_X(x)x\right)\left(\int dy\, f_Y(y)y\right) = \overline{X}\,\overline{Y}$$

2- Uncorrelated R.V's are not necessarily independent: independent $\Rightarrow$ uncorrelated, but uncorrelated might or might not be independent.

3- An exception of 2- above is for X and Y jointly Gaussian. Then independent $\Leftrightarrow$ uncorrelated.

Theorem: If X and Y are uncorrelated, and $Z = X + Y$, then $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$.

Proof: Since $E[XY] = E[X]E[Y]$, and $\sigma_X^2 = E[(X-\overline{X})^2] = E[X^2] - \overline{X}^2 = \overline{X^2} - \overline{X}^2$ ... then...

Central Moments:

$\overline{X} = E[X]$, k<u>th</u> central moment $\equiv E[(X-\overline{X})^k] = \begin{cases} \int_{-\infty}^{\infty} dx\ (x-\overline{X})^k f_X(x) \\ \sum_i (x_i - \overline{X})^k p_X(x_i) \end{cases}$. 1st central moment = 0, 2nd central moment = σ_X^2

$$\sigma_X^2 = E[(X-\overline{X})^2] = E[X^2] - 2\overline{X}E[X] + \overline{X}^2 = E[X^2] - \overline{X}^2 = \overline{X^2} - \overline{X}^2$$

Examples:

I. Binomial R.V.

$p_X(n) = \dfrac{N!}{(N-n)!n!} p^n (1-p)^{N-n}$ $\boxed{\overline{K} = Np}$, $\overline{K^2} = N(N-1)p^2 + Np$,

$$\sigma_K^2 = \overline{K^2} - \overline{K}^2 = N(N-1)p^2 + Np - N^2p^2 = Np - Np^2 = Np(1-p)$$

II. Poisson R.V.

$p_N(n) = \sum \dfrac{1}{n!}(\lambda T)^n e^{-\lambda T}$, $\boxed{\overline{N} = \lambda T}$, $\overline{N^2} = (\lambda T)^2 + \lambda T$, $\boxed{\sigma_N^2 = \overline{N^2} - \overline{N}^2 = \lambda T}$

III. Exponential R.V.

$f_X(x) = \lambda e^{-\lambda x} u(x)$ $\boxed{\overline{X} = \dfrac{1}{\lambda}}$, $\boxed{\overline{X^2} = 2/\lambda^2}$, $\boxed{\sigma_X^2 = \dfrac{2}{\lambda^2} - \dfrac{1}{\lambda^2} = \dfrac{1}{\lambda^2}}$

IV. Uniform

V. Gaussian

$\boxed{\overline{X} = m}$, $\boxed{\overline{X^2} = \sigma^2 + m^2}$, $\boxed{\sigma_X^2 = \sigma^2}$

$$Np + N(N-1)p^2 \sum_{n=0}^{N-2} \frac{(N-2)!}{(N-2-n)!n!} p^n (1-p)^{N-2-n} = Np + N(N-1)p^2 \text{ (binomial theorem again)}$$

$$\boxed{E[K^2] = N(N-1)p^2 + Np}$$

II. Poisson R.V.

$$p_N(n) = \sum \frac{1}{n!}(\lambda T)^n e^{-\lambda T} \qquad E[N] = \lambda T \qquad \boxed{E[N^2] = (\lambda T)^2 + \lambda T} \text{ Excercise (optional)}$$

Continuous Examples:

I. Uniform R.V.

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0, & \text{else} \end{cases} \qquad E[X^k] = \frac{1}{k+1}. \qquad \text{For } f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0, & \text{else} \end{cases},$$

$$E[X^k] = \int_{-\infty}^{\infty} x^k f_X(x) dx = \int_a^b \left(\frac{1}{b-a}\right) x^k dx = \left(\frac{1}{b-a}\right) \frac{x^{k+1}}{k+1}\bigg|_a^b = \frac{b^{k+1} - a^{k+1}}{(b-a)(k+1)}$$

II. Exponential R.V. $E[X^2] = 2/\lambda^2$

III. Gaussian R.V.

$$E[X^2] = \sigma^2 + m^2$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}, \ E[X] = m. \ E[X^2] = \int_{-\infty}^{\infty} dx \frac{x^2}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$$ **. Let** $\zeta = x - m \Rightarrow d\zeta = dx$

Then $$E[X^2] = \int_{-\infty}^{\infty} d\zeta \frac{m^2 + 2\zeta m + \zeta^2}{\sqrt{2\pi\sigma^2}} e^{-\zeta^2/2\sigma^2} = m^2 + 0 + \int_{-\infty}^{\infty} d\zeta \frac{\zeta^2}{\sqrt{2\pi\sigma^2}} e^{-\zeta^2/2\sigma^2} = m^2 + \sigma^2$$

Moments:

The k^th^ moment of R.V. $X \equiv E[X^k] = \overline{X^k} = \langle X^k \rangle$.

For ***discrete R.V.*** $\overline{X^k} = \sum_i x_i^k p_X(x_i)$, and for ***continuous R.V.*** $\overline{X^k} = \int_{-\infty}^{\infty} dx\, x^k f_X(x)$

$\bar{X} =$ first moment or mean, $\overline{X^2} =$ second moment. Sometimes such a quantity is called “a statistic.”

Discrete Examples:

I. Binomial R.V.

$p_K(k) = \frac{N!}{(N-k)!k!} p^k (1-p)^{N-k}$ = probability of k successes in N independent trials, where p = probability of success in one trial.

$$E[K] = Np$$

$$E[K^2] = \sum_{k=0}^{N} k^2 p_K(k) = \sum_{k=0}^{N} \frac{k^2 N!}{(N-k)!k!} p^k (1-p)^{N-k}$$. Let $k^2 = k(k-1) + k$, then

$$E[K^2] = \sum_{k=0}^{N} \frac{k(k-1)N!}{(N-k)!k!} p^k (1-p)^{N-k} + \sum_{k=0}^{N} \frac{kN!}{(N-k)!k!} p^k (1-p)^{N-k} = \sum_{k=2} + \sum_{k=1}$$

$$E[K^2] = \sum_{k=2}^{N} \frac{N!}{(N-k)!(k-2)!} p^k (1-p)^{N-k} + \sum_{k=1}^{N} \frac{N!}{(N-k)!(k-1)!} p^k (1-p)^{N-k} = \sum_{k=2}^{N} + E[K]]$$

Let $n = k-2$, then

$$E[K^2] = Np + \sum_{n=0}^{N-2} \frac{N!}{(N-n-2)!n!} p^{n+2} (1-p)^{N-n-2} =$$

Continuous Examples:

For continuous R.V. $E[X] = \int_{-\infty}^{\infty} dx\ x f_X(x)$

I. Exponential R.V.

$f_X(x) = \begin{cases} 0 & ;\ x < 0 \\ \lambda e^{-\lambda x} & ;\ x \ge 0 \end{cases}$ $E[X] = \int_{-\infty}^{\infty} dx\ x f_X(x) = \int_0^{\infty} dx\ x\lambda e^{-\lambda x} = \frac{1}{\lambda}$ i.e. $\boxed{\overline{X} = \frac{1}{\lambda}}$, which occurs at

$x = \frac{\lambda}{e}$.

In obtaining above result, we used integration by parts: $\int_0^{\infty} u dv = uv\big|_0^{\infty} - \int_0^{\infty} v du$. Let $u = \lambda x$, and $v = -\frac{1}{\lambda} e^{-\lambda x}$

II. Gaussian R.V.

$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$. $E[X] = \int_{-\infty}^{\infty} dx\ x f_X(x) = \int_{-\infty}^{\infty} dx \frac{x}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$. Let $\zeta = x - m \Rightarrow d\zeta = dx$

Then $E[X] = \int_{-\infty}^{\infty} d\zeta \frac{m+\zeta}{\sqrt{2\pi\sigma^2}} e^{-\zeta^2/2\sigma^2} = m \int_{-\infty}^{\infty} d\zeta \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\zeta^2/2\sigma^2} + \int_{-\infty}^{\infty} d\zeta \frac{\zeta}{\sqrt{2\pi\sigma^2}} e^{-\zeta^2/2\sigma^2} = m$

$$E[X] = \sum_{n=0}^{N} n p_X(n) = \sum_{n=0}^{N} \frac{n(N!)}{(N-n)!n!} p^n (1-p)^{N-n} = \sum_{n=1}^{N} \frac{n(N!)}{(N-n)!n!} p^n (1-p)^{N-n} =$$

$\sum_{n=1}^{N} \frac{N!}{(N-n)!(n-1)!} p^n (1-p)^{N-n}$. substitute $k = n-1$, then

$$\sum_{k=0}^{N-1} \frac{N!}{(N-k-1)!k!} p^{k+1} (1-p)^{N-k-1} = Np \sum_{k=0}^{N-1} \frac{(N-1)!}{((N-1)-k)!k!} p^k (1-p)^{(N-1)-k}$$

$= Np(p+1-p)^{N-1} = Np$

Recall: *Binomial Theorem:* $(a+b)^N = \sum_{n=0}^{N} \binom{N}{n} a^n b^{N-n}$

IV. Poisson R.V.

Experiment: observe the times of emissions of photoelectrons from a photocathode. $S = \{(t_1, t_2, t_3, \ldots): 0 \le t_1 < t_2 < t_3 < \ldots\}$.

Define R.V. $N =$ number of electrons emitted in time interval $(0, T)$.

Poisson pmf: (physical model) $p_N(n) = \frac{1}{n!}(\lambda T)^n e^{-\lambda T}$.

In this case, λ is a parameter dependent on light intensity. Suppose $\lambda T = 5$, then:

n	0	1	2	3	4	5	6	7	8	9	10
$p_N(n)$	0.0067	0.0337	0.084	0.14	0.175	0.175	0.146	0.104	0.0652	0.036	0.018

$$E[N] = \sum_{n=0}^{\infty} \frac{n}{n!}(\lambda T)^n e^{-\lambda T} = e^{-\lambda T} \sum_{n=1}^{\infty} \frac{n}{n!}(\lambda T)^n = \lambda T e^{-\lambda T} \sum_{n=1}^{\infty} \frac{1}{(n-1)!}(\lambda T)^{n-1} = \lambda T e^{-\lambda T} \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda T)^k$$

Then, $E[N] = \lambda T e^{-\lambda T} e^{\lambda T} = \lambda T$

Recall: $e^x = \sum_{k=0}^{\infty} x^k / k!$, near $x = 0$. (Taylor series)

Statistical Averages: Examples

Discrete Examples:

For discrete R.V. with pmf $p_X(x_i) = P\{X = x_i\}$, $E[X] = \sum_i x_i p_X(x_i)$, a weighted average of sample values.

I. Bernoulli R.V.

Experiment: “Bernoulli Trial” $S = \{\text{Success}, \text{Failure}\} = \{s, f\}$. $P(\{s\}) = p$, $P(\{f\}) = 1 - p$.

Bernoulli R.V. $X(\{\text{Success}\}) = 1$, $X(\{\text{Failure}\}) = 0$. $p_X(1) = p$, $p_X(0) = 1 - p$.

$$E[X] = \sum_i x_i p_X(x_i) = 1 p_X(1) + 0\ p_X(0) = p$$

II. Dice

Toss two dice: $S = \{(n, m) : 1 \le n, m \le 6\}$, define $X((n, m)) = n + m$.

$$E[X] = \sum_i x_i p_X(x_i) = 2p_X(2) + 3p_X(3) + \ldots + 12p_X(12) = 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + \ldots + 12\left(\frac{1}{36}\right) = 7$$

III. Binomial R.V.

Experiment: N independent Bernoulli trials. $S = \{(s, s, \ldots, s), (s, s, \ldots, s, f), \ldots, (f, f, \ldots, f)\}$. 2^N sample points. Define the ***Binomial R. V.*** by mapping each sample point into an integer (subset of reals) equal to the

number of successes. How many points are there with n successes and N-n failures? N!/n!(N-n)! Therefore $p_X(n) = \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} = \binom{N}{n} p^n (1-p)^{N-n}$, where $\binom{N}{n} \equiv C(N, n) = \frac{N!}{(N-n)!n!}$.

Statistical Averages (Mathematical Expectation)

First moment, $E[X] = \int_{-\infty}^{\infty} dx\ x f_X(x)$, k<u>th</u> moment $E[X^k] = \int_{-\infty}^{\infty} dx\ x^k f_X(x)$; $E[g(X)] = \int_{-\infty}^{\infty} dx\ g(x) f_X(x)$.

Given $Z = g(X, Y)$ and $f_{X, Y}(x, y)$, find $E[Z]$:

Method 1: $E[Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{X, Y}(x, y) dx dy$ *Method 2: Find* $f_Z(z)$, *and then* $E[Z] = \int_{-\infty}^{\infty} dz\ z f_Z(z)$

Linearity Property of Statistical Average:

$$E[g_1(X) + g_2(X)] = \int_{-\infty}^{\infty} dx\ (g_1(x) + g_2(x)) f_X(x) = E[g_1(X)] + E[g_2(X)],$$

$$E[g_1(X, Y) + g_2(X, Y)] = E[g_1(X, Y)] + E[g_2(X, Y)].$$

To specify deviation from mean, consider $X - \bar{X}$, where $\bar{X} = E[X]$:

$E[X - \bar{X}] = 0$, not very useful.
$E[|X - \bar{X}|]$, works but awkward.

$\sigma_X^2 = E[(X - \bar{X})^2]$, Variance of X. Generally used to measure deviation from mean.

$\sigma_X = \sqrt{E[(X - \bar{X})^2]}$, Standard Deviation.

Central Moments: $\bar{X} = E[X]$, k<u>th</u> central moment $\equiv E[(X - \bar{X})^k] = \begin{cases} \int_{-\infty}^{\infty} dx\ (x - \bar{X})^k f_X(x) \\ \sum_i (x_i - \bar{X})^k p_X(x_i) \end{cases}$.

1st central moment = 0, 2nd central moment = σ_X^2

$$\sigma_X^2 = E[(X - \bar{X})^2] = E[X^2] - 2\bar{X}E[X] + \bar{X}^2 = E[X^2] - \bar{X}^2 = \overline{X^2} - \bar{X}^2$$