$$\phi_Z(j\omega) = \left(\phi_X(0) + \omega\phi'_X(0) + \frac{\omega^2}{2}\phi''_X(0) + \ldots\right)^n = \left(1 - \frac{\omega^2}{2}\sigma_X^2 + \ldots\right)^n \approx \left(1 - \frac{\omega^2}{2}\sigma_X^2\right)^n$$

Then $\phi_Z(j\omega) \approx \left(1 - \frac{\omega^2}{2}\frac{\sigma_Z^2}{n}\right)^n$. Recall $\lim_{n \to \infty}\left(1 + \frac{\alpha}{n}\right)^n = e^{\alpha}$, then $\boxed{\lim_{n \to \infty}\phi_Z(j\omega) \approx e^{-\frac{\omega^2}{2}\sigma_Z^2}}$,

which is characteristic function of Gaussian with zero mean, and variance of $\sigma_Z^2 = n\sigma_X^2$.

A Central Limit Theorem

If $X_1, X_2, \ldots, X_n$ are **i**ndependent, **i**dentically **d**istributed (i.i.d.) R.V's with mean m_X and variance σ_X^2,

and $Z = \sum_i X_i$, then $\lim_{n \to \infty} F_Z(z) = \Phi\left(\frac{z - nm_X}{\sqrt{n}\sigma_X}\right)$.

which is a C.D.F. of a Gaussian with mean nm_X and variance $n\sigma_X^2$.

For a continuous R.V, $\lim_{n \to \infty} f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x - nm_X)^2 / 2n\sigma_X^2}$. For discrete...

Example: Sum of i.i.d. R.V's with uniform pdf

$$f_X(x) = \begin{cases} \frac{1}{2a} & -a \le x \le a \\ 0, & \text{else} \end{cases} \quad . \; \overline{X} = m_X = 0, \; \sigma_X^2 = \frac{a^2}{3}.$$

$$\boxed{\phi_X(j\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} dx\, e^{j\omega x} f_X(x) = \int_{-a}^{a} dx\, \frac{e^{j\omega x}}{2a} = \frac{1}{2a} \frac{e^{j\omega a} - e^{-j\omega a}}{j\omega} = \frac{\sin \omega a}{\omega a}}.$$

Let $Z = \sum_{i=1}^{n} X_i$, $\overline{Z} = n\overline{X} = 0$, $\sigma_Z^2 = n\sigma_X^2 = n\frac{a^2}{3}$, and $\phi_Z(j\omega) = \left(\frac{\sin \omega a}{\omega a}\right)^n$

Since $\phi_Z(j\omega)$ is concentrated near origin, then use Taylor series approximation:

$$\phi_X(j\omega) = \phi_X(0) + \omega\phi'_X(0) + \frac{\omega^2}{2}\phi''_X(0) + \ldots,$$

III. Poisson R.V

$$p_K(k) = \frac{1}{k!}\mu^k e^{-\mu}, \quad k = 0, 1, 2, \ldots$$

$$\phi_K(j\omega) = E[e^{j\omega K}] = \sum_{k=0}^{\infty} e^{j\omega k} p_K(k) = \sum_{k=0}^{\infty} e^{j\omega k}\frac{1}{k!}\mu^k e^{-\mu} = \left(\sum_{k=0}^{\infty}\frac{(\mu e^{j\omega})^k}{k!}\right)e^{-\mu} = e^{(\mu e^{j\omega})}e^{-\mu}$$

$$\phi_K(j\omega) = e^{\mu(e^{j\omega}-1)}.$$

As a check: $\phi_K(0) = 1$

Sum of two Poisson:

Let K_1 and K_2 be two independent Poisson R.V's: $p_{K_1}(k) = \frac{1}{k!}\mu^k e^{-\mu}$, and $p_{K_2}(k) = \frac{1}{k!}\lambda^k e^{-\lambda}$.

Let $N = K_1 + K_2$. Then $\phi_N(j\omega) = \phi_{K_1}(j\omega)\phi_{K_2}(j\omega) = e^{\mu(e^{j\omega}-1)}e^{\lambda(e^{j\omega}-1)} = e^{(\mu+\lambda)(e^{j\omega}-1)}$.

Therefore, *N* is Poisson with $p_N(n) = \frac{(\mu+\lambda)^k}{n!}e^{-(\mu+\lambda)}$

2- Moment Generation

Since $\phi_X(j\omega) = \int_{-\infty}^{\infty} dx\ e^{j\omega x} f_X(x)$, then $\phi_X(0) = \int_{-\infty}^{\infty} dx\ f_X(x) = 1$.

Also $\frac{d}{d\omega}\phi_X(j\omega) = \phi'_X(\omega) = \int_{-\infty}^{\infty} dx\ jxe^{j\omega x} f_X(x)$. Then $\phi'_X(0) = j\int_{-\infty}^{\infty} dx\ x f_X(x) = j\overline{X}$.

Similarly, $\phi''_X(0) = -\overline{X^2}$.

In general, $\phi_X^{(n)}(j\omega) = \int_{-\infty}^{\infty} dx\ (jx)^n e^{j\omega x} f_X(x)$, resulting in $\phi_X^{(n)}(0) = j^n\int_{-\infty}^{\infty} dx\ x^n f_X(x) = j^n\overline{X^n}$.

So: $\boxed{\overline{X^n} = \frac{1}{j^n}\phi_X^{(n)}(0)}$.

Examples

I. Exponential pdf

$\boxed{f_X(x) = \lambda e^{-\lambda x} u(x)}$, $\phi_X(j\omega) = \int_{-\infty}^{\infty} dx\ e^{j\omega x} f_X(x) = \int_0^{\infty} dx\ \lambda e^{(j\omega - \lambda)x} = \frac{\lambda}{j\omega - \lambda} e^{(j\omega - \lambda)x}\Big|_0^{\infty} = \frac{\lambda}{\lambda - j\omega}$

The nth derivative $\phi_X^{(n)}(j\omega) = n!(j)^n \lambda(\lambda - j\omega)^{-(n+1)}$, when evaluated at $\omega = 0$, results in

$\boxed{E[X^n] = \frac{1}{j^n}\phi_X^{(n)}(0) = \frac{n!}{\lambda^n}}$.

II. Gaussian pdf

$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}}$ which results in $\boxed{\phi_X(j\omega) = e^{j\omega m} e^{-\omega^2\sigma^2/2}}$

Application: Sum of two independent Gaussian R.V's

Characteristic Functions

Definition:

Characteristic function of *R.V. X* is defined by: $\boxed{\phi_X(j\omega) = E[e^{j\omega X}]}$. Where $\phi_X(j\omega)$ is a function of real parameter ω, and is defined for all real values of ω.

For X discrete; $\phi_X(j\omega) = \sum_i e^{j\omega x_i} p_X(x_i)$.

For X continuous; $\phi_X(j\omega) = \int_{-\infty}^{\infty} dx\, e^{j\omega x} f_X(x)$, which is the Fourier transform of $f_X(x)$.

Continuous Case

$$\phi_X(j\omega) = F\{f_X(x)\} = \int_{-\infty}^{\infty} dx\, e^{j\omega x} f_X(x)$$

$$f_X(x) = F^{-1}\{\phi_X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-j\omega x} \phi_X(j\omega)$$

Use of Characteristic Functions

1- **Sums of R.V's:** Let X and Y be independent R.V's, and let $Z = X + Y$, then

$$f_Z(z) = \int_{-\infty}^{\infty} d\zeta\, f_X(z-\zeta) f_Y(\zeta), \text{ and}$$

$$\phi_Z(j\omega) = E[e^{j\omega Z}] = E[e^{j\omega(X+Y)}] = E[e^{j\omega X} e^{j\omega Y}] = E[e^{j\omega X}]E[e^{j\omega Y}] = \phi_X(j\omega)\phi_Y(j\omega)$$

Generalize: If $Z = \sum_i X_i$, all independent R.V's, then $\phi_Z(j\omega) = \prod_{i=1}^{n} E[e^{j\omega X_i}] = \prod_{i=1}^{n} \phi_{X_i}(j\omega)$