EE 631: Estimation and Detection
Part 6

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Representation of Signals

Deterministic Signals

$$\begin{aligned} x_1(t) &= A\cos\omega_0(t) \\ x_2(t) &= A\sin\omega_0(t) \end{aligned}$$

We are interested in expressing these signals in terms of a set of orthonormal basis functions, i.e.

$$x(t) = \sum_{i=1}^{\infty} x_i \phi_i(t)$$

$\{\phi_i(t)\}$ are the orthonormal basis functions.

$$\begin{aligned} \underbrace{<\phi_i, \phi_j>}_{\text{Inner product}} &\triangleq \int_T \phi_i(t)\phi_j^*(t)dt \\ &= \delta_{ij} \\ &= \begin{cases} 1 & ; i = j \quad \text{(implies power is normalized to 1)} \\ 0 & ; i \neq j \quad \text{(implies orthogonality)} \end{cases} \end{aligned}$$

The main purpose of representing signals via a linear combination of orthogonal signals is that they can be easily added with and subtracted from each other as in vector operations.
The coefficients x_i's are the "projections" of $x(t)$ into $\phi_i(t)$s.

$$\begin{aligned} <x, \phi_j> &= \text{projection of } x(t) \text{ onto } \phi_j(t) \\ &= \int_T x(t)\phi_j^*(t)dt \\ &= \int_T \sum_i x_i \phi_i(t)\phi_j^*(t)dt \\ &= \sum x_i \int_T \underbrace{\phi_i(t)\phi_j^*(t)dt}_{=\delta_{ij}} \\ &= \sum_{i=1}^{\infty} x_i \delta_{ij} \\ &= x_j \\ \Rightarrow x_j &= <x, \phi_j> \end{aligned}$$

e.g. If $x(t)$ is a periodic signal with period T, then

$$\phi_n(t) = \frac{1}{\sqrt{T}} \exp(jn\omega_0 t)$$

where $\omega_0 = \frac{2\pi}{T}$, $x(t) = \sum_{n=-\infty}^{\infty} x_n \phi_n(t)$ and $x_n = \frac{1}{\sqrt{T}} \int_T x(t) \exp(jn\omega_0 t) dt$. Energy of $x(t)$:

$$\begin{aligned} E_x &= \int_T |x(t)|^2 dt \\ &= \int_T x(t) x^*(t) dt \\ &= \int_T \sum_n x_n \phi_n(t) \sum_m x_m^* \phi_m^*(t) dt \\ &= \sum_n \sum_m x_n x_m^* \underbrace{\int_T \phi_n(t) \phi_m^*(t) dt}_{\delta_{mn}} \\ \Rightarrow E_x &= \sum_n |x_n|^2 = \int_T |x(t)| dt \end{aligned}$$

This is known as the *Parseval's theorem.*
If $x(t)$ is approximately represented via a finite sum:

$$x(t) \approx x_N(t)$$

where

$$x_N(t) = \sum_{n=1}^{N} x_n \phi_n(t)$$

then, $E_{X_N} = \sum_{n=1}^{N} |x_n|^2 \leq E_x = \sum_{n=1}^{\infty} |x_n|^2$.
If $\{\phi_i(t)\}$ is such that the energy goes to zero as $N \to \infty$, i.e.

$$\lim_{N \to \infty} (E_x - E_{x_N}) = 0$$

then $\{\phi_i(t)\}$ is said to be a Complete OrthoNormal (CON) set.
e.g.: The harmonics $\phi_n(t) = \frac{1}{\sqrt{T}} \exp(jn\omega_0 t)$ where $t \in [0, T]$ and $n = 0, \pm 1, \pm 2, ... \pm \infty$, form a CON set.

Evaluation of x_i

Evaluation of x_i using the correlator and matched filter are shown in figures 1 and 2, respectively.

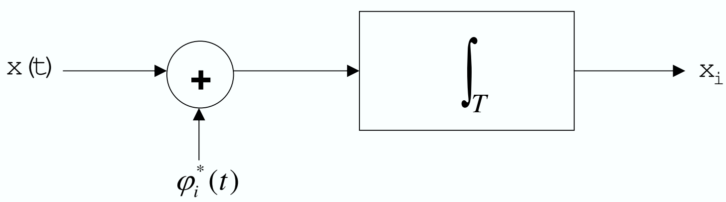

Figure 1. Correlator implementation

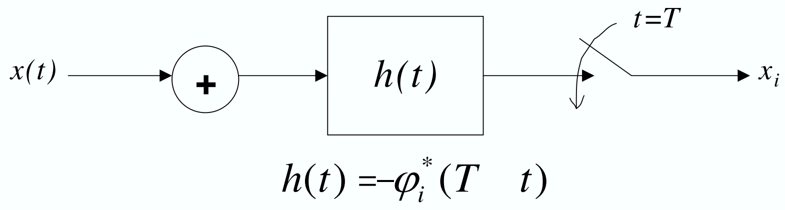

Figure 2. Matched filter implementation

<u>Correlation of two signals</u>
Correlation between two signals is defined via:

$$\rho_{xy} = \frac{E_{xy}}{\sqrt{E_x E_y}} = \frac{< x, y >}{\sqrt{< x, x >< y, y >}}$$

where

$$\begin{aligned}
E_{xy} &=< y, x > \\
&= \textstyle\int_T \sum y_n \phi_n(t). \sum x_m^* \phi_m^*(t) dt \\
&= \textstyle\sum_{n,m} y_n x_m^* \int_T \phi_n(t) \phi_m^*(t) dt \\
&= \textstyle\sum_{n,m} y_n x_m^* \delta_{mn} \\
\Rightarrow E_{xy} &= \textstyle\sum_n y_n x_n^*
\end{aligned}$$

<u>Gram Schimdt Procedure</u>
Given a set of M signals $s_i(t)$, $i = 1, 2, ..., M$
We are interested in finding a set of orthonormal basis functions $\phi_i(t)$, $i = 1, 2, ..., N \leq M$ that span the linear signal subspace of $s_i(t)$, i.e.
if

$$p(t) = \sum_{i=1}^{M} \alpha_i s_i(t)$$

where α_i's are constants, then

$$p(t) = \sum_{i=1}^{N} p_i \phi_i(t)$$

where

$$p_i =< p, \phi_i >$$

However, in general

$$\alpha_i \neq< p, s_i >$$

A procedure for constructing the $\phi_i(t)$'s, called the *Gram Schmidt Proceudre*, is described next.

Algorithm:

1. Let
$$\psi_1(t) \triangleq s_1(t)$$
and
$$E_{\psi_1} =< \psi_1, \psi_1 >$$
Define
$$\phi_1(t) = \frac{\psi_1(t)}{\sqrt{E_{\psi_1}}}$$
Note that $< \phi_1, \phi_1 >= E_{\phi_1} = 1$.

2. Let
$$\psi_2(t) = s_2(t) - \underbrace{< s_2, \phi_1 > \phi_1(t)}_{\text{projection of } s_2(t) \text{ on to } \phi_1(t)}$$
Note that:
$$\begin{aligned} < \psi_2, \phi_1 > &=< s_2, \phi_1 > - \underbrace{< s_2, \phi_1 >}_{constant} < \phi_1, \phi_1 > \\ &=< s_2, \phi_1 > - < s_2, \phi_1 > \\ &= 0 \end{aligned}$$
i.e $\psi_2 \perp \phi_1$. Again define:
$$\phi_2(t) = \frac{\psi_2(t)}{\sqrt{E_{\psi_2}}}$$
Note that $< \phi_2, \phi_1 >= 0$.

3. Let
$$\begin{aligned} \psi_3(t) &= s_3(t) - < s_3, \phi_1 > \phi_1(t) - < s_3, \phi_2 > \phi_2(t) \\ &= s_3(t) - \textstyle\sum_{i=1}^{2} < s_3, \phi_i > \phi_i(t) \end{aligned}$$
Define
$$\phi_3(t) = \frac{\psi_3(t)}{\sqrt{E_{\psi_3}}}$$
Note that $< \phi_3, \phi_i >= 0$ for $i = 1, 2$. i.e. $\phi_3 \perp \phi_i$.

4. Continuing like this for k steps we get:
$$\psi_k(t) = s_k(t) - \sum_{i=1}^{k-1} < s_k, \phi_i > \phi_i(t)$$
and
$$\phi_k(t) = \frac{\psi_k(t)}{\sqrt{E_{\psi_k}}}$$
where $< \phi_k, \phi_i >= 0$ for $i = 1, 2, ..., k-1$. i.e. $\phi_k \perp \phi_i$.

This procedure is repeated with all $s_k(t)$'s that yield a non-zero residual i.e. $\psi_k(t)$. If the residual is zero then just skip that $s_k(t)$. This implies that $s_k(t)$ is linearly dependent on $s_i(t)$, for $i = 1, 2, ..., k-1$.

Outcome: A set of orthonormal basis functions $\Phi = \{\phi_i(t); i = 1, 2, ..., N \leq M\}$

Matrix representation:

$$\vec{s}(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_N(t) \end{bmatrix} \qquad \vec{\phi}(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \vdots \\ \phi_N(t) \end{bmatrix}$$

$$\vec{A} = \begin{bmatrix} <s_1,\phi_1> & 0 & 0 & \cdots & 0 \\ <s_2,\phi_1> & <s_2,\phi_2> & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ \vdots & & & \ddots & \vdots \\ <s_N,\phi_1> & & \cdots & & <s_N,\phi_N> \end{bmatrix}$$

$$\vec{S}(t) = \vec{A}\vec{\phi}(t)$$

Example:

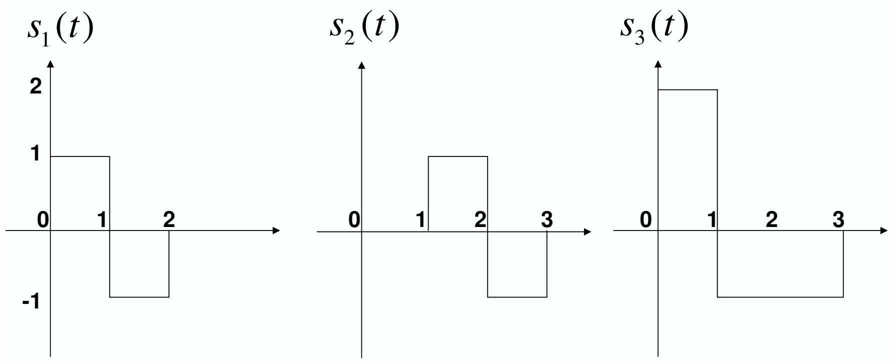

Figure 3. Example

1. Begin by:

$$\psi_1(t) = s_1(t)$$
$$E_{\psi_1} = 2$$

2. Compute

$$< s_1, \phi_1 >= \int_0^3 s_2(t)\frac{s_1(t)}{\sqrt{2}}dt = \frac{-1}{2}$$
$$\psi_2(t) = s_2(t) - \left(\frac{-1}{\sqrt{2}}\right)\frac{s_1(t)}{\sqrt{2}} = s_2(t) + \frac{s_1(t)}{2}$$
$$E_{\psi_2}(t) = \frac{3}{2} \Rightarrow \phi_2(t) = \sqrt{\frac{2}{3}}\left[s_2(t) + \frac{s_1(t)}{2}\right]$$

3. By observation, we can write that

$$\begin{aligned} s_3(t) &= 2s_1(t) + s_2(t) \\ &= \left[s_2(t) + \tfrac{s_1(t)}{2}\right] + \tfrac{3}{2}s_1(t) \\ &= \psi_2(t) + \tfrac{3}{2}\psi_1(t) \\ &= \sqrt{\tfrac{3}{2}}\phi_2(t) + \tfrac{3\sqrt{2}}{2}\phi_1(t) \end{aligned}$$

Hence two basis functions are sufficient.

Summary:

$$\begin{aligned}
s_1(t) &= \sqrt{2}\phi_1(t) \\
s_2(t) &= -\tfrac{1}{2}\phi_1(t) + \sqrt{\tfrac{3}{2}}\phi_2(t) \\
s_3(t) &= \tfrac{3}{\sqrt{2}}\phi_1(t) + \sqrt{\tfrac{3}{2}}\phi_2(t) \\
\vec{A} &= \begin{bmatrix} \sqrt{2} & 0 \\ -\frac{1}{2} & \sqrt{\frac{3}{2}} \\ \frac{3}{\sqrt{2}} & \sqrt{\frac{3}{2}} \end{bmatrix}
\end{aligned}$$

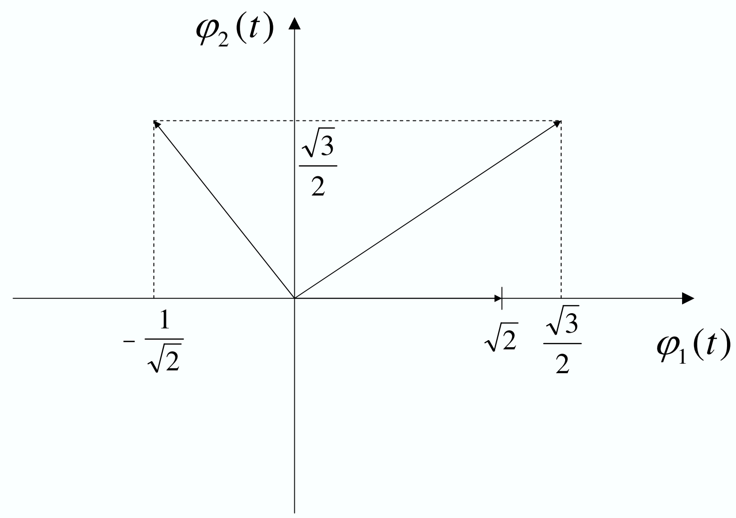

Figure 4.

Some common signal sets

1. Antipodal signals

$$s_1(t) = -s_2(t)$$

i.e. 180° out of phase with each other.

$$E_s = < s_1, s_1 > = < s_2, s_2 >$$

Metric distance

$$d \triangleq 2\sqrt{E_s}$$
$$s_1(t) = \sqrt{E_s}\phi_1(t)$$
$$s_2(t) = -\sqrt{E_s}\phi_2(t)$$

e.g.

$$\begin{aligned} s_1(t) &= A\cos(\omega_0 t) \\ s_2(t) &= -A\cos(\omega_0 t) \end{aligned}$$

where $0 \leq t \leq T$

$$\begin{aligned} \Rightarrow \phi_1(t) &= \sqrt{\tfrac{2}{T}}\cos(\omega_0 t) \\ E_s &= \tfrac{A^2 T}{2} \end{aligned}$$

This corresponds to antipodal ASK signaling or BPSK.

2. Orthogonal signals

$$< s_1, s_2 >= 0$$

e.g.

$$\begin{aligned} s_1(t) &= \cos(\omega_0 t) \\ s_2(t) &= \sin(\omega_0 t) \end{aligned}$$

where $0 \leq t \leq T$. Therefore,

$$\begin{aligned} \phi_1(t) &= \sqrt{\tfrac{2}{T}}\cos(\omega_0 t) \\ \phi_2(t) &= \sqrt{\tfrac{2}{T}}\sin(\omega_0 t) \end{aligned}$$

e.g. PSK where the two phases are 90° out of phase with each other.

Random signals

Series representation of random signals is called the *Karhunen Loeve* (KL) transform.
Let $\{\phi)i(t); i = 1, 2, ..., \infty\}$ be a set of orthonormal (deterministic) basis functions.
Consider any sample function $x(t)$ of a random process and express it in terms of $\{\phi_i(t)\}$ where $t \in [0, T]$.

$$x(t) = \sum_{i=1}^{\infty} x_i \phi_i(t)$$

The coefficients x_i's are random variables. For simplicity, we assume $E(x_i) = 0$.
We are interested in selecting $\{\phi_i(t)\}$ such that x_i's are uncorrelated/orthogonal.

$$\begin{aligned} E[(x_i - \bar{x_i})(x_j - \bar{x_j}^*)] &= E(x_i x_j^*) = \begin{cases} \sigma_i^2 & ; i = j \\ 0 & ; i \neq j \end{cases} \\ &= \sigma_i^2 \delta_{ij} \end{aligned}$$

Consider the KL representation:

$$x(t) = \sum_{i=1}^{\infty} x_i \phi_i(t)$$

We multiply both sides by with x_j^*

$$x_j^* x(t) = x_j^* \sum_{i=1}^{\infty} x_i \phi_i(t)$$

Take expectation on both sides

$$E[x_j^* x(t)] = \sum_{i=1}^{\infty} E[x_i x_j^*] \phi_i(t) \tag{1}$$

We showed earlier that the coefficients of orthonormal expansion of a signal are obtained via:

$$\begin{aligned} x_j &= \int_T x(u)\phi_j^*(u)du \\ x_j^* &= \int_T x*(u)\phi_j(u)du \end{aligned} \tag{2}$$

Substitute 2 in 1 to yield:

$$E[\int_T x^*(u)\phi_j(u)du x(t)] = \sum_{i=1}^{\infty} E(x_i x_j^*)\phi_i(t)$$

We wish to impose the following condition on the above equation:

$$E(x_i x_j^*) = \lambda_i \delta_{ij}$$

This yields:

$$\begin{aligned} \int_T E\big[x(t)x^*(u)\big]\phi_j(u)du &= \lambda_j \phi_j(t) \\ \int_T R_x(t,u)\phi_j(u)du &= \lambda_j \phi_j(t) \end{aligned}$$

This equation is the solution for:

$$\boxed{\begin{array}{rll} \text{eigenfunctions :} & \phi_j(t) & j = 1, 2, ... \\ \text{eigenvalues :} & \lambda_j & j = 1, 2, ... \end{array}}$$

for the autocorrelation function.

Thus, to represent a stochastic process via a linear combination of orthonormal basis functions with orthogonal coefficients, the basis functions should be chosen to be the eigen functions of the autocorrelation of the random signal. The variance of the ith coefficient σ_i^2 is the ith eigenvalue λ_i^2 of the autocorrelation function.

The eigenfunction domain is the generalized spectral domain for the stochastic process. The generalized spectral representation provides a signal processing framework to analyze/address various detection and estimation problems.