Parameter Estimation:

Type ①: MMSE estimator:

$$\hat{A}_{MMSE}(\underline{R}) = E\{A/\underline{R}\}$$

Type ②: Minimum Absolute value of error (MAP) Estimator

$$\hat{A}_{abs}(\underline{R}) = \text{median of } \{A/\underline{R}\}$$

Type ③: MAP (Maximum A Posteriori) Estimator

$$\hat{A}_{MAP} = \text{argmax} \ln p(A/\underline{R})$$

Type ④: ML (Maximum Likelihood) Estimator

$$\hat{A}_{ML}(\underline{R}) = \text{argmax } L\{A\} \leftarrow \text{likelihood function}$$
$$= \text{argmax } p(\underline{R}; A)$$

① The observation Y is given by $Y = X + N$, where X, N are r.v's. N is normal with mean one & variance σ^2, X is uniformly distributed over $[0, 2]$. Determine the MAP estimate of the parameter $\hat{X}$.

Soln:

$$Y = X + N$$
$$P_N(N) \sim N(\phi, \sigma^2)$$
$$P_X(X) = \begin{cases} 1/2, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

to find $p(X|Y)$:

$$p(X|Y) = \frac{p(X) \cdot p(Y|X)}{p(Y)}$$

$$p(Y|X) = P_N(Y-X) \sim N(Y-X, \sigma^2)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(Y-X-1)^2}{2\sigma^2}\right\}$$

$$\hat{X}_{MAP} = \underset{X}{\operatorname{argmax}} \left\{ \ln p(Y|X) + \ln p(X) \right\}$$

$$\Rightarrow \frac{\partial}{\partial x} \ln p(Y|X) + \frac{\partial}{\partial x} \ln p(X) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left\{-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(Y-X-1)^2}{2\sigma^2}\right\} + \frac{\partial}{\partial x} \ln p(X) = 0$$

$$\Rightarrow -\frac{1}{2\sigma^2} \cdot 2(Y-X-1)(-1) = 0$$

$$\Rightarrow \boxed{\hat{X}_{MAP} = Y-1}$$

(2). Find $\hat{x}_{MS}$, from the observations,

$$Y = X + N$$

X & N are r.v's. with p.d.f's.

$$P_X(X) = \frac{1}{2}\left[\delta(x-1) + \delta(x+1)\right] \quad \& \quad P_N(N) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp -\left(\frac{N^2}{2\sigma^2}\right)$$

Soln: $\hat{X}_{MS} = E\{p(X|Y)\}$

$$p(X|Y) = \frac{P_X(X) \cdot p(Y|X)}{P_Y(Y)}$$

$$p(Y|X) = P_N(Y-X) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(Y-X)^2}{2\sigma^2}\right]$$

$$P_Y(Y) = \int_{-\infty}^{\infty} p(Y|X) \cdot P_X(X)\, dX$$

$$P_Y(Y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(Y-X)^2}{2\sigma^2}\right] \cdot \underbrace{\frac{1}{2}\left(\delta(x-1)+\delta(x+1)\right)}_{\text{zero except for } x = 1,-1} \cdot dX$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(Y-X)^2}{2\sigma^2}\right\} \cdot \frac{1}{2}\Big|_{x=-1} + \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{(Y-X)^2}{2\sigma^2}\right\} \cdot \frac{1}{2}\Big|_{x=1}$$

$$= \frac{1}{2\sqrt{2\pi\sigma^2}}\left[\exp\left\{-\frac{(Y+1)^2}{2\sigma^2}\right\} + \exp\left\{-\frac{(Y-1)^2}{2\sigma^2}\right\}\right]$$

$$\hat{X}_{MS} = E\left[P(X|Y)\right]$$

$$= \int_{-\infty}^{\infty} X \cdot P(X|Y)\, dX$$

$$= \int_{-\infty}^{\infty} X \cdot \frac{\exp\left[-\frac{(Y-X)^2}{2\sigma^2}\right] \cdot \left[\delta(x-1)+\delta(x+1)\right]}{\exp\left[-\frac{(Y+1)^2}{2\sigma^2}\right] + \exp\left[-\frac{(Y-1)^2}{2\sigma^2}\right]} \cdot dX$$

$$= \frac{(-1)\exp\left[-\frac{(Y+1)^2}{2\sigma^2}\right]}{\exp\left[\frac{-(Y+1)^2}{2\sigma^2}\right] + \exp\left[\frac{-(Y-1)^2}{2\sigma^2}\right]} + \frac{(+1)\exp\left[-\frac{(Y-1)^2}{2\sigma^2}\right]}{\exp\left[-\frac{(Y+1)^2}{2\sigma^2}\right] + \exp\left[-\frac{(Y-1)^2}{2\sigma^2}\right]}$$

$$= \frac{e^{\frac{y}{\sigma^2}} - e^{-y/\sigma^2}}{e^{y/\sigma^2} + e^{-y/\sigma^2}} = \frac{1 - e^{-2y/\sigma^2}}{1 + e^{-2y/\sigma^2}}$$

③. We make K observations $R_1, R_2 .. R_K$

where $r_i = a + n_i$; $a \sim N(0, \sigma_a^2)$

$n_i \sim N(0, \sigma_n^2)$

(i) Find the MMSE estimate, $\hat{a}_{MS}$.

$$\hat{a}_{MS} = E[p(a|\vec{r})]$$

$$p(a|\vec{r}) = \frac{p(a) \cdot p(\vec{r}|a)}{p(\vec{r})}$$

$$p(a) \cdot p(\vec{r}|a) = \frac{1}{\sqrt{2\pi\sigma_a^2}} \cdot \exp\left(\frac{-a^2}{2\sigma_a^2}\right) \cdot \prod_{i=1}^{K} \frac{1}{\sqrt{2\pi\sigma_n^2}} \cdot \exp\left\{\frac{-(r_i - a)^2}{2\sigma_n^2}\right\}$$

$$= \frac{1}{\sqrt{2\pi\sigma_a^2}} \left(\frac{1}{\sqrt{2\pi\sigma_n^2}}\right)^K \exp\left\{\underbrace{\frac{-a^2}{2\sigma_a^2} - \sum_{i=1}^{K} \frac{(a - r_i)^2}{2\sigma_n^2}}\right\}$$

$$\frac{a^2\sigma_n^2 + Ka^2\sigma_a^2 + \sigma_a^2 \sum r_i^2 - 2(\sigma_a^2 \sum r_i)a}{2\sigma_a^2\sigma_n^2}$$

$$= \frac{a^2(\sigma_n^2 + K\sigma_a^2) - 2(\sigma_a^2 \sum r_i)a + \sigma_a^2 \sum r_i^2}{2\sigma_a^2\sigma_n^2}$$

$$= \frac{a^2 - \frac{2(\sigma_a^2 \sum r_i)a}{\sigma_n^2 + K\sigma_a^2} + \frac{\sigma_a^2 \sum r_i^2}{\sigma_n^2 + K\sigma_a^2}}{\frac{2\sigma_a^2\sigma_n^2}{\sigma_n^2 + K\sigma_a^2}}$$

$$= \frac{a^2 - \frac{2(\sigma_a^2 \sum r_i)a}{\sigma_n^2 + K\sigma_a^2} + \overbrace{\left[\frac{\sigma_a^2 \sum r_i}{\sigma_n^2 + K\sigma_a^2}\right]^2}^{\beta} + \left[\frac{\sigma_a^2 \sum r_i^2}{\sigma_n^2 + K\sigma_a^2} - \beta\right]}{}$$

$$= \left[a - \frac{\sigma_a^2 \sum r_i}{\sigma_n^2 + K\sigma_a^2}\right]^2 \Big/ \frac{2\sigma_a^2\sigma_n^2}{\sigma_n^2 + K\sigma_a^2}$$

$$p(a|\vec{r}) = \underbrace{\frac{\frac{1}{\sqrt{2\pi\sigma_a^2}}\left[\frac{1}{\sqrt{2\pi\sigma_n^2}}\right]^k}{p(\vec{r})}}_{①} \cdot \underbrace{\exp\left\{-\frac{\left(a - \frac{\sigma_a^2 \sum r_i}{\sigma_n^2 + k\sigma_a^2}\right)^2}{\frac{2\sigma_a^2\sigma_n^2}{\sigma_n^2 + k\sigma_a^2}}\right\}}_{②}$$

① constant, $\vec{r}$ is given

② gaussian : mean: $\frac{\sigma_a^2 \sum r_i}{\sigma_a^2 + k\sigma_a^2}$

var $= \frac{\sigma_a^2 \sigma_n^2}{\sigma_n^2 + k\sigma_a^2}$

$$\hat{a}_{MS} = E[p(a|\vec{r})]$$
$$= \frac{\sigma_a^2 \sum r_i}{\sigma_n^2 + k\sigma_a^2}$$

(ii) Find the MAP estimate $\hat{a}_{MAP}$.

$$\hat{a}_{MAP} = \underset{a}{\operatorname{argmax}}\ p(a|\vec{r})$$

$\because p(a|\vec{r})$ is Gaussian p.d.f.

when $a = E\{a|\vec{r}\} \Rightarrow p(a|\vec{r})$ is maximized.

$$\Rightarrow \hat{a}_{MAP} = \hat{a}_{MS} = \frac{\sigma_a^2 \sum r_i}{\sigma_n^2 + k\sigma_a^2}$$

(iii) Compute the mean-squared error:

$$MSE = E[(\hat{A}_{MS} - A)^2]$$
$$= E[\hat{A}_{MS}^2 + A^2 - 2\hat{A}_{MS} \cdot A]$$

$$\hat{A}_{MS} = \hat{a}_{MS} \sim N(0, \sigma_a^2)$$

$$E[\hat{A}_{MS}^2] = \hat{A}_{MS}^2, \quad E(A) = 0, \quad E(A^2) = \sigma_a^2$$

$$MSE = \frac{\sigma_a^2 \sum r_i}{\sigma_n^2 + K\sigma_a^2} + \sigma_a^2$$

(4). let $Y_1, Y_2 \ldots Y_K$ be observed random variables such that

$$Y_k = a + bx_k + Z_k \quad , k = 1, \ldots K$$

The constants x_k, $k = 1, \ldots K$ are known, while a & b are unknown constants. The r.v's. Z_k, $k = 1, 2 \ldots K$ are statistically independent each with zero mean & variance σ^2. Obtain the ML estimate of (a, b).

Soln:

$$Y_k = a + bx_k + Z_k$$

$Z_k \sim N(0, \sigma^2)$, $k = 1, 2 \ldots K$ (independent r.v's)

x_k - constants (known)

a, b - constants (unknown)

$Y_k \sim N(a + bx_k, \sigma^2)$, $k = 1, 2 \ldots K$.

Y_k's are statistically independent.

$$P_Y(Y) = P(Y; a, b)$$

$= L(a, b) \rightarrow$ likelihood function.

$$= \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{ \frac{-(Y_k - (a + bx_k))^2}{2\sigma^2} \right\}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^K \exp\left\{ \frac{-1}{2\sigma^2} \sum_{k=1}^{K} [y_k - a - bx_k]^2 \right\}$$

$$\ln(L(a,b)) = -K\ln\sigma - \frac{K}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum(y_k - a - bx_k)^2$$

$$\hat{a}_{ML} = \underset{a}{\arg\max}\left(\ln L(a,b)\right)$$

$$\Rightarrow \frac{\partial}{\partial a}\ln L(a,b) = \frac{\partial}{\partial b}\ln L(a,b) = 0$$

$$-\frac{1}{2\sigma^2}\sum 2(y_k - a - bx_k)(-1) = 0 \quad ; \quad -\frac{1}{2\sigma^2}\sum 2(y_k - a - bx_k).(-x_k) = 0$$

$$\Rightarrow \sum(y_k - a - bx_k) = 0 \quad ; \quad \sum(x_k y_k - a\,x_k - b\,x_k^2) = 0$$

$$\therefore a = \frac{1}{K}\sum y_k - \frac{b}{K}\sum x_k \quad ;$$

$$\sum x_k y_k - \left(\frac{1}{K}\sum y_k - \frac{b}{K}\sum x_k\right)\sum x_k - b\sum x_k^2 = 0$$

$$\Rightarrow \sum x_k y_k - \frac{1}{K}\sum x_k \sum y_k + b\left\{\frac{1}{K}\sum x_k . \sum x_k - \sum x_k^2\right\} = 0$$

$$\therefore \hat{b}_{ML} = \frac{\sum x_k y_k - \frac{1}{K}\sum x_k \sum y_k}{\underbrace{\sum x_k^2 - \frac{1}{K}\sum x_k . \sum x_k}}$$

$$\hat{a}_{ML} = \frac{1}{K}\sum y_k - \frac{1}{K}\left[\qquad\qquad\right]\sum x_k$$