(1) $H_1: r = bm_1 + n$

$H_0: r = n$

b & n → independent r.v's, m_1 → constant

$b \sim N(0, \sigma_b^2)$

$n \sim N(0, \sigma_n^2)$

Find the L.R.T.

Soln: $H_0: r = n$

$$p(r|H_0) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left[\frac{-r^2}{2\sigma_n^2}\right]$$

$H_1: r = bm_1 + n$

$$p(r|H_1) = (0, \sigma_1^2) \qquad \sigma_1^2 = m_1^2\sigma_b^2 + \sigma_n^2$$

LRT : $$\Lambda(\underline{R}) = \frac{p(\underline{R}|H_0)}{p(\underline{R}|H_1)}$$

$$= \frac{\frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{r^2}{2\sigma_0^2}\right)}{\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(\frac{-r^2}{2\sigma_1^2}\right)} \qquad (\sigma_0^2 = \sigma_n^2)$$

$$= \frac{\sigma_0}{\sigma_1} \exp\left(\frac{r^2}{2\sigma_0^2} - \frac{r^2}{2\sigma_1^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\ln\frac{\sigma_0}{\sigma_1} + \frac{r^2}{2\sigma_0^2} - \frac{r^2}{2\sigma_1^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\eta)$$

$$r^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\left(\ln(\eta) + \ln\left(\frac{\sigma_1}{\sigma_0}\right)\right)\left(\frac{1}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}}\right)$$

(2) let, $R = S + N$

S, N - independent r.v's

$N \sim N(0, \sigma_n^2)$

H_0: $S \sim N(\mu_0, \sigma_s^2)$

H_1: S takes the value of μ_{11} & μ_{12} with probabilities P_{11} & P_{12} $(= 1 - P_{11})$

construct L.R.T.

Soln: H_0: $S \sim N(\mu_0, \sigma_s^2)$, $N \sim (0, \sigma_n^2)$

$$R \sim N(\mu_0, \underbrace{\sigma_s^2 + \sigma_n^2}_{\sigma_0^2})$$

$$H_1: R = \begin{cases} \mu_{11} + N & , P_{11} \\ \mu_{12} + N & , P_{12} = 1 - P_{11} \end{cases}$$

$$p(R|H_0) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{(r-\mu_0)^2}{2\sigma_0^2}\right\}$$

$$p(R|H_1) = P_{11} \cdot \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left\{-\frac{(r-\mu_{11})^2}{2\sigma_n^2}\right\}$$

$$+ P_{12} \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left\{-\frac{(r-\mu_{12})^2}{2\sigma_n^2}\right\}$$

(3) H_0: $r = -2 + n$

H_1: $r = 4 + n$

$n \sim N(-1, \sigma^2)$

(i) Construct LRT with $P_0 = 0.3$, Unitary costs for false decision & zero cost for the right decision.

(ii) What is the probability of error?

Soln: H_0: $r = -2 + n$, $r \sim N(-3, \sigma^2)$

H_1: $r = 4 + n$, $r \sim N(3, \sigma^2)$

$$p(r|H_0) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{(r+3)^2}{2\sigma^2}\right\}$$

$$p(r|H_1) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left\{-\frac{(r-3)^2}{2\sigma_0^2}\right\}$$

LRT. $\Lambda(R) = \dfrac{\exp\left[\dfrac{-(r-3)^2}{2\sigma_d^2}\right]}{\exp\left\{-\dfrac{(r+3)^2}{2\sigma_d^2}\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \dfrac{P_0}{P_1} \dfrac{C_{10}-C_{00}}{C_{01}-C_{11}}$

$P_0 = 0.3, \; P_1 = 0.7, \; C_{10} = C_{01} = 1$

$C_{00} = C_{11} = 0$

$\Rightarrow \dfrac{(r+3)^2 - (r-3)^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\dfrac{3}{7}\right)$

$\Rightarrow r \underset{H_0}{\overset{H_1}{\gtrless}} \dfrac{\sigma^2}{6} \ln\left(\dfrac{3}{7}\right) \triangleq \alpha$

$P_e = P_0 \, Pr(\text{choosing } H_1 | H_0) + P_1 \, Pr(\text{choosing } H_0 | H_1)$

$= 0.3 \, . \, Pr(-2+n > \alpha) + 0.7 \, Pr(4+n < \alpha)$

(4) $\Lambda(R) = \dfrac{p(R|H_1)}{p(R|H_0)} \quad (= \Lambda)$

prove that (1) $E\{\Lambda^n | H_1) = E(\Lambda^{n+1} | H_0)$

Soln: For LHS :

$E\{\Lambda^n | H_1) = \int_Z \Lambda^n . p(R|H_1) . d\underline{R}$

$\left[E(X) = \int X . p(x) \, dX, \; E(X|Y) = \int X . p(X|Y) \, dX \right]$

$= \int_Z \left[\dfrac{p(R|H_1)}{p(\underline{R}|H_0)}\right]^n . \; p(R|H_1) \, d\underline{R}$

For RHS:

$$E(\Lambda^{n+1}|H_0) = \int_Z \Lambda^{n+1} \cdot p(\underline{R}|H_0)\, d\underline{R}$$

$$= \int_Z \left[\frac{p(\underline{R}|H_1)}{p(\underline{R}|H_0)}\right]^{n+1} \cdot p(\underline{R}|H_0)\, d\underline{R}$$

$$= \int_Z \left[\frac{p(\underline{R}|H_1)}{p(\underline{R}|H_0)}\right]^{n} \cdot p(\underline{R}|H_1)\, d\underline{R}$$

$$\therefore\ E(\Lambda^n|H_1) = E(\Lambda^{n+1}|H_0)$$

(ii) $E(\Lambda|H_1) - E(\Lambda|H_0) = var(\Lambda|H_0)$ – to prove.

Soln: $var(\Lambda|H_0) = \underbrace{E((\Lambda|H_0)^2)}_{\downarrow} - E^2(\Lambda|H_0)$

$$= E(\Lambda|H_1) - E^2(\Lambda|H_0)$$

$$E(\Lambda|H_0) = \int_Z \Lambda \cdot p(\underline{R}|H_0)\, d\underline{R}$$

$$= \int_Z \frac{p(\underline{R}|H_1)}{p(\underline{R}|H_0)} \cdot p(\underline{R}|H_0)\, d\underline{R}$$

$$= \int_Z p(\underline{R}|H_1) \cdot d\underline{R}$$

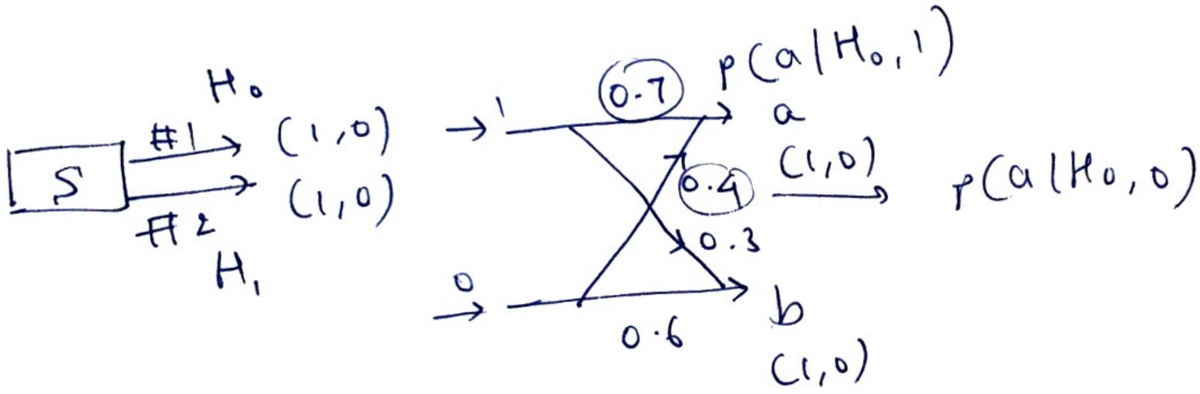
$\Rightarrow \underbrace{E(\Lambda|H_0) = 1 = E^2(\Lambda|H_0)}$

$\Rightarrow E^2(\Lambda|H_0) = 1$

$$var(\Lambda|H_0) = E(\Lambda|H_1) - E(\Lambda|H_0)$$

(4) One of the two possible sources supply the inputs to the simple communication channel. Both sources put out either 0 or 1. The numbers on the line are the channel transition probabilities, given by

$$Pr(a\,out \mid 1\,in) = 0.7$$



(1) False alarm : say source 2 when source 1 is detected

(2) Detection, say source 2 when source 2 is detected.

Q1. Compute the ROC of a test that maximizes P_D subject to the constraint that $P_F = \alpha$

Q2. Describe the test procedure in detail for $\alpha = 0.25$.

Soln: Source ①: $P(1) = P(0) = 0.5$

$$p(1, H_0) = p(0, H_0) = 0.5$$

Source ②: $p(1, H_1) = 0.6$, $p(0, H_1) = 0.4$.

$$\Lambda(R) = \frac{p(r|H_1)}{p(r|H_0)}$$

$$= \begin{cases} \dfrac{p(a|H_1)}{p(a|H_0)}, & r = a \\[2ex] \dfrac{p(b|H_1)}{p(b|H_0)}, & r = b \end{cases}$$

$$p(a|H_0) = \underline{p(a|H_0, 1)} \cdot p(1, H_0) + p(a|H_0, 0) \cdot p(0, H_0)$$

$$= 0.7 \times 0.5 + 0.4 \times 0.5$$

$$= 0.55$$

$$p(b|H_0) = 1 - p(a|H_0)$$

$$= 0.45$$

$$p(a|H_1) = p(a|H_1, 1) \cdot p(1, H_1) + p(a|H_1, 0) \cdot p(0, H_1)$$

$$= 0.7 \times 0.6 + 0.4 \times 0.4$$

$$= 0.58$$

$$p(b|H_1) = 1 - p(a|H_1) = 0.42$$

$$\Lambda(R) = \begin{cases} \dfrac{p(a|H_1)}{p(a|H_0)} = \dfrac{0.58}{0.55} = 1.05, & r = a \\ \dfrac{p(b|H_1)}{p(b|H_0)} = \dfrac{0.42}{0.45} = 0.93, & r = b. \end{cases}$$

① $R < 0.93$, we choose H_0

② $R > 1.05$, we choose H_1

③ $0.93 \leq R \leq 1.05$, $\underline{P_D}$ & $\underline{\underline{P_F}}$??

let $p(H_0|a) = p(\text{choose } H_0 | a \text{ is received})$

$$= P_{a0}$$

$$p(H_0|b) = P_{b0}$$

$$\Rightarrow \quad p(H_1|a) = 1 - P_{a0}$$

$$p(H_1|b) = 1 - P_{b0}$$

$P_F = p(\text{choosing } H_1 | H_0 \text{ is sent})$

$$= p(H_1|a) \cdot p(a|H_0) + p(H_1|b) \cdot p(b|H_0)$$

$$= (1 - P_{a0})\, 0.55 + (1 - P_{b0}) \cdot 0.45$$

$$P_F = 1 - 0.55 P_{a0} - 0.45 P_{b0} = \alpha$$

$P_D = p(\text{choose } H_1 | H_1 \text{ is sent})$

$$= p(H_1|a) \cdot p(a|H_1) + p(H_1|b) \cdot p(b|H_1)$$

$$= (1 - P_{a0}) \cdot 0.58 + (1 - P_{b0}) \cdot 0.42$$

$$= 1 - 0.58 P_{a0} - 0.42 P_{b0}$$

$$P_D = \underbrace{P_F} - 0.03 P_{a0} + 0.03 P_{b0}$$

$$= \alpha + 0.03(P_{b0} - P_{a0})$$

$$\underline{P_D = 0.25 + 0.03(P_{b0} - P_{a0})}$$

P_D can be maximized if $(P_{b0} - P_{a0})$ is maximized.

$$P_{a0} = p(H_0|a)$$

$$P_{b0} = p(H_0|b)$$