$X(t)$ & $Y(t)$ are zero-mean random processes & uncorrelated.

Autocorrelation:

$$K_x(t,u) = f_1(t)\, f_1(u)$$

$$K_Y(t,u) = f_2(t)\, f_2(u)$$

where $f_1(t) = \cos t$; $t \in [0, 2\pi]$

$f_2(t) = \sin t$

let $z(t) = x(t) + Y(t) + N(t)$

$$K_n(t,u) = 0.5\, \delta(t-u)$$

Find the eigen values & eigen functions of $K_z(t,u)$.

<u>Soln</u>: Since $X(t)$, $Y(t)$ & $N(t)$ are uncorrelated random processes.

$$K_z(t,u) = E\{z(t).z^*(u)\}$$

$$= E\{[X(t) + Y(t) + N(t)].[X(u) + Y(u) + N(u)]^*\}$$

$$= E[X(t).X^*(u)] + E[Y(t).Y^*(u)] + E[N(t).N^*(u)]$$

$$= K_X(t,u) + K_Y(t,u) + K_n(t,u)$$

{ <u>Reminder</u>: separable kernels, i.e.

$$R_X(t,u) = f(t).f^*(u)$$

where $f(t)$ is deterministic, then, $\phi_1(t) = \frac{f(t)}{\sqrt{E_f}}$

$$\lambda_1 = E_f$$ }

$$\phi_1(t) = \frac{f_1(t)}{\sqrt{E_{f_1}}} \quad \& \quad \phi_2(t) = \frac{f_2(t)}{\sqrt{E_{f_2}}}$$

$$E_{f_1} = \int_0^{2\pi} |f_1(t)|^2 dt \quad ; \quad E_{f_2} = \int_0^{2\pi} |f_2(t)|^2 dt$$

$$= \int_0^{2\pi} \cos^2 t \, dt \qquad = \int_0^{2\pi} \sin^2 t \, dt$$

$$= \int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right) dt \qquad \underline{E_{f_2} = \pi}$$

$$= \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$\underline{E_{f_1} = \frac{1}{2}[2\pi] = \pi}$$

to prove that $\phi_1(t)$ & $\phi_2(t)$ are orthogonal.

$$\langle f_1, f_2 \rangle = \int_0^{2\pi} f_1(t). f_2^*(t) dt$$

$$= \int_0^{2\pi} \cos t . \sin t \, dt$$

$$= \int_0^{2\pi} \frac{1}{2} \sin 2t \, dt$$

$$= \frac{1}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= 0$$

$$f_1(t) \perp f_2(t) \Rightarrow \underline{\phi_1(t) \perp \phi_2(t)}$$

The Eigen functions:

$$\phi_1(t) = \frac{1}{\sqrt{\pi}} \cos t \, , \quad \phi_2(t) = \frac{1}{\sqrt{\pi}} \sin t$$

$$K_z(t,u) = K_x(t,u) + K_y(t,u) + K_n(t,u)$$

$$= f_1(t) f_1(u) + f_2(t) f_2(u) + 0.5\,\delta(t-u)$$

$$= \sqrt{\pi}\,\phi_1(t) . \sqrt{\pi}\,\phi_1(u) + \sqrt{\pi}\,\phi_2(t) . \sqrt{\pi} . \phi_2(u) + 0.5\,\delta(t-u)$$

$$= \pi\,\phi_1(t)\,\phi_1(u) + \pi\,\phi_2(t)\,\phi_2(u) + 0.5\,\delta(t-u)$$

for white noise $N(t)$,

$$K_n(t,u) = \frac{N_0}{2}\,\delta(t-u)$$

$$\lambda_i = \frac{N_0}{2}$$

& ϕ_i = any constant for all i

$$\delta(t-u) = \sum_i \phi_i(t)\,\phi_i^*(u)$$

also, we know that

if $r(t) = x(t) + n(t) \leftarrow$ white noise,

then $\lambda_{r_i} = \lambda_{x_i} + \frac{N_0}{2}$

$\Rightarrow$ $\lambda_1 = \pi + 0.5$, $\lambda_2 = \pi + 0.5$ } Eigen values

$\lambda_i = 0.5$ for $i = 3, 4, 5, \ldots$

$\phi_1(t) = \frac{\cos t}{\sqrt{\pi}}$, $\phi_2 = \frac{\sin(t)}{\sqrt{\pi}}$ } Eigen functions

ϕ_i = costant for $i = 3, 4, 5, \ldots \infty$