LRT :- $\Lambda(\underline{R}) = \dfrac{p(\underline{R} \mid H_1)}{p(\underline{R} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \dfrac{P_0 (C_{10} - C_{00})}{P_1 (C_{01} - C_{11})}$

$$P(X, Y) = p(X)\, p(Y \mid X)$$
$$= p(Y) \cdot p(X \mid Y)$$

$$p(X) = \int p(X \mid Y)\, p(Y)\, dY$$

$$E\{X\} = \int X\, p(X)\, dX$$

$$\mathrm{var}(X) = E\{X^2\} - E^2\{X\}$$

$$E\{\Lambda^n \mid H\} = \int_Z \Lambda^n\, p(\underline{R} \mid H_1)\, d\underline{R}$$

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1. given: $H_1 : r = s + n$

$H_0 : r = n$

$s, n \rightarrow r.v$

$$P_s(s) = \begin{cases} a \cdot e^{-as} & s \geq 0 \end{cases}$$

$$P_n(N) = \begin{cases} b \cdot e^{-bN} & , N \geq 0 \end{cases}$$

prove that LRT reduces to $R \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$ (gamma)

Soln: $p(r|H_1)$ & $p(r|H_0)$,

for H_1, $r = s + n$

$$p(r|H_1) = P_s(S) * P_n(N) \Big|_{s=r, N=r}$$

$$= \int_{-\infty}^{\infty} P_s(\tau) \cdot P_N(r-\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} a \cdot e^{-a\tau} \cdot b \cdot e^{-b(r-\tau)} \, d\tau$$

$$= a \cdot b e^{-br} \int_{-\infty}^{\infty} e^{(b-a)\tau} \, d\tau$$

$$= a \cdot b \cdot e^{-br} \int_{0}^{r} e^{(b-a)\tau} \, d\tau$$

$$= \frac{ab}{b-a} \left(e^{-ar} - e^{-br} \right), \quad r \geq 0$$

$$p(r|H_0) = P_n(r) \qquad H_0 : r = n$$

$$= b \cdot e^{-br}$$

LRT : $$\Lambda(r) = \frac{p(r|H_1)}{p(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$= \underbrace{\frac{a}{b-a}\left[e^{-(a-b)r} - 1\right]}_{} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

to prove : $$R \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

$$e^{-(a-b)r} - 1 \underset{H_0}{\overset{H_1}{\gtrless}} \eta\left(\frac{b-a}{a}\right)$$

$$(b-a)r \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left\{\eta\left(\frac{b-a}{a}\right)\right\}$$

$$\Rightarrow \underline{R} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{b-a} \ln\left[\left(\frac{b-a}{a}\right)\eta + 1\right] = \gamma$$

Here $$\eta = \frac{P_0(C_{10} - C_{00})}{P_1(C_{01} - C_{11})}$$

(2) $H_1: \ P_r(R) = \frac{1}{2} \exp(-|R|)$

$H_0: \ P_r(R) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} R^2\right)$

Soln:

$$\Lambda(R) = \frac{p(R|H_1)}{p(R|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\frac{\frac{1}{2} \exp(-|R|)}{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} R^2\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\Rightarrow \sqrt{\frac{\pi}{2}} \exp\left(-|R| + \frac{1}{2} R^2\right) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\Rightarrow -|R| + \frac{1}{2} R^2 \underset{H_0}{\overset{H_1}{\gtrless}} \ln \sqrt{\frac{2}{\pi}} \eta$$

if $R > 0$, $\quad -R \frac{1}{2} R^2 \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\eta) - \ln \sqrt{\frac{\pi}{2}}$

if $R < 0$, $\quad R + \frac{1}{2} R^2 \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\eta) - \ln \sqrt{\frac{\pi}{2}}$

(3) Suppose, $X \sim N(m, b^2)$, this is passed through one of the non-linear transforms,

$H_0: \ Y = X^2$

$H_1: \ Y = e^X$

$p(Y|H_1)$ & $p(Y|H_0)$

$H_0 : Y = X^2$

$\Rightarrow X_1 = -\sqrt{Y}$ & $X_2 = +\sqrt{Y}$, $Y > 0$

$$P_y(y) = \frac{P_X(X)}{|Y'|}\Bigg|_{x=f(y)}$$

$$Y' = 2X$$

$$P_X(X) = \frac{1}{\sqrt{2\pi b^2}}\left\{\exp\left(\frac{-(\sqrt{y}-m)^2}{2b^2}\right) + \exp\left(\frac{-(-\sqrt{y}-m)^2}{2b^2}\right)\right\}$$

$$\downarrow$$

$$P_Y(y) = \frac{1}{\sqrt{2\pi b^2}}\frac{1}{|2X|}\left\{ \right\}$$

$$= \frac{1}{\sqrt{2\pi b^2}}\frac{1}{2\sqrt{Y}}\left\{\exp\frac{-(+\sqrt{y}-m)^2}{2b^2} + \exp\left(\frac{-(-\sqrt{Y}-m)^2}{2b^2}\right)\right\}$$

$H_1 : Y = e^X$, $\Rightarrow X = \ln Y$

$$\underline{Y'} = e^X = \underline{Y}$$

$$\therefore P_Y(Y) = \frac{P_X(X)}{|Y'|}$$

$$= \frac{1}{\sqrt{2\pi b^2}}\exp\left[-\frac{(\ln Y - m)^2}{2b^2}\right]\cdot\frac{1}{|Y|}$$

$$\left\{P(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{-(x-m)^2}{2\sigma^2}\right]\right\}$$

$$\Lambda(R) = \frac{p(Y|H_1)}{p(Y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

④ $H_0 : r_i \sim N(0, b_0^2)$

$H_1 : r_i \sim N(0, b_1^2)$; r_i are indepedent observations

$i = 1, 2 \ldots k$ & $b_0 < b_1$

Soln: let $i = 1, 2 \ldots k$

& $j = 0, 1$

$p(\underline{R}|H_j)$, $j = 0, 1$

$$p(\underline{R}|H_j) = \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi b_j^2}} \exp\left\{-\frac{r_i^2}{2b_j^2}\right\}$$

$$\Lambda(\underline{R}) = \frac{p(\underline{R}|H_1)}{p(\underline{R}|H_0)}$$

$$= \frac{\prod_{i=1}^{k} \frac{1}{\sqrt{2\pi b_1^2}} \exp\left\{\frac{-r_i^2}{2b_1^2}\right\}}{\prod_{i=1}^{k} \frac{1}{\sqrt{2\pi b_0^2}} \exp\left\{\frac{-r_i^2}{2b_0^2}\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\Rightarrow \left(\frac{b_0}{b_1}\right)^k \exp\left[-\frac{1}{2}\left(\frac{1}{b_1^2} - \frac{1}{b_0^2}\right)\sum_{i=1}^{k} r_i^2\right] \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\Rightarrow \underline{\underline{\sum_{i=1}^{k} r_i^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \left(\frac{\ln(\eta) - k\ln\left(\frac{b_0}{b_1}\right)}{b_1^2 - b_0^2}\right) = \gamma$$