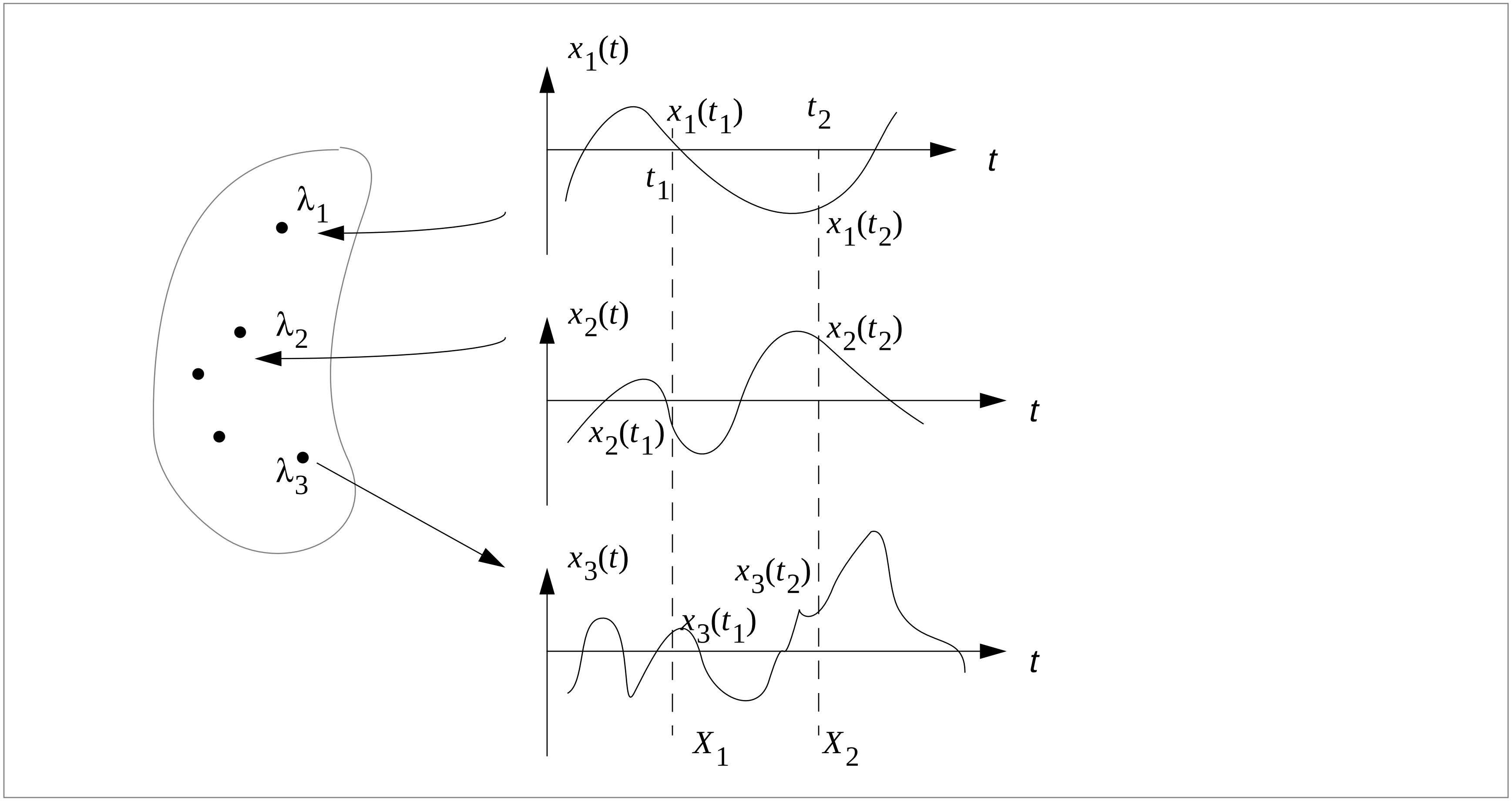
Random Processes

Definitions:

A *random process* is a family of random variables indexed by a parameter $t \in T$, where T is called the i*ndex set*.

Experiment outcome is λ_i, which is a whole function $X(t, \lambda_i) = x_i(t)$. this real-valued function is called a *sample function*. The set of all sample functions is an *ensamble*.



Statistics of Random Processes

I. Distributions and Densities

Random process $X(t)$. For a particular value of t, say t_1, we have a random variable $X(t_1) = X_1$. The distribution function of this random variable is defined by

$F_X(x_1;t_1) = P\{X(t_1) \le x_1\}$, and is called the *first-order* distribution of $X(t)$.

The corresponding *first-order* density function is $f_X(x_1;t_1) = \frac{\partial}{\partial x_1} F_X(x_1;t_1)$.

For t_1 and t_2, we get two random variables $X(t_1) = X_1$ and $X(t_2) = X_2$. Their joint distribution is called the *second-order* distribution:

$F_X(x_1, x_2;t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$, with corresponding second-order density function

$$f_X(x_1, x_2;t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2;t_1, t_2).$$

The nth-order distribution and density functions are given by

$F_X(x_1, x_2, \ldots x_n;t_1, t_2, \ldots, t_n) = P\{X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_n) \le x_n\}$, and

$$f_X(x_1, x_2, \ldots x_n;t_1, t_2, \ldots, t_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} F_X(x_1, x_2, \ldots x_n;t_1, t_2, \ldots, t_n).$$

II. Statistical Averages (ensamble averages)

The ***mean*** of $X(t)$ is defined by $E[X(t)] = \bar{X}(t) = \int_{-\infty}^{\infty} x f_X(x;t) dx$.

The ***autocorrelation*** of $X(t)$ is defined by

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2) dx_1 dx_2 .$$

The ***autocovariances*** of $X(t)$ is defined by

$$C_{XX}(t_1, t_2) = E\{[X(t_1) - \bar{X}(t_1)][X(t_2) - \bar{X}(t_2)]\} = R_{XX}(t_1, t_2) - \bar{X}(t_1)\bar{X}(t_2)$$

The ***nth joint moment*** of $X(t)$ is defined by

$$E[X(t_1)X(t_2)\ldots X(t_n)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1 x_2 \ldots x_n f_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) dx_1 dx_2 \ldots dx_n$$

III. Stationarity

Strict-Sense Stationarity

A random process $X(t)$ is called ***strict-sense stationary (SSS)*** if the statistics are invariant w.r.t. any time shift, i.e. $f_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = f_X(x_1, x_2, \ldots, x_n; t_1 + c, t_2 + c, \ldots, t_n + c)$

It follows $f_X(x_1; t_1) = f_X(x_1; t_1 + c)$ for any c ,hence first-order density of a SSS $X(t)$ is independent of time t: $f_X(x_1; t) = f_X(x_1)$. Similarly, $f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + c, t_2 + c)$.

By setting $c = -t_1$, we get $f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_2 - t_1)$. Thus, if $X(t)$ is SSS, the joint density of the random variables $X(t)$ and $X(t + \tau)$ is independent of t and depends only on the time difference τ.

Wide -Sense Stationary

A random process $X(t)$ is said to be ***wide-sense stationary (WSS)*** if its mean is constant (independent of time) $E[X(t)] = \bar{X}$, and its autocorrelation depends only on the time difference τ. $E[X(t)X(t+\tau)] = R_{XX}(\tau)$

As a result, the auto covariance of a WSS process also depends only on the time difference τ:

$C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2$.

Setting $\tau = 0$ in $E[X(t)X(t+\tau)] = R_{XX}(\tau)$ results in $E[X^2(t)] = R_{XX}(0)$. The average power of a WSS process is independent of time t, and equals $R_{XX}(0)$.

An SSS process is WSS, but a WSS process is not necessarily SSS.

Two processes $X(t)$ and $Y(t)$ are jointly wide-sense stationary (jointly WSS) if each is WSS and their cross-correlation depends only on the time difference τ:

$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau)$ Also the cross-covariance of jointly WSS $X(t)$ and $Y(t)$ depends only on the time difference τ:

$C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\bar{Y}$.

IV. Time Averages and Ergodicity

The ***time-averaged mean*** of a sample function $x(t)$ of a random process $X(t)$ is defined as

$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt$, where the symbol $\langle . \rangle$ denotes *time-averaging*.

Similarly, the ***time-averaged autocorrelation*** of the sample function $x(t)$ is

$$\langle x(t)x(t+\tau)\rangle = \lim_{T \to \infty} \frac{1}{T}\int_{-T/2} x(t)x(t+\tau)dt\,.$$

Both $\langle x(t)\rangle$ and $\langle x(t)x(t+\tau)\rangle$ are random variables since they depend on which sample function resulted from experiment. then if $X(t)$ is stationary:

$$E[\langle x(t)\rangle] = \lim_{T \to \infty} \frac{1}{T}\int_{-T/2}^{T/2} E[x(t)]dt = \bar{X}\,.$$

The expected value of the time-averaged mean is equal to ensamble mean.

Also $E[\langle x(t)x(t+\tau)\rangle] = \lim_{T \to \infty} \frac{1}{T}\int_{-T/2}^{T/2} E[x(t)x(t+\tau)]dt = R_{XX}(\tau)$,

the expected value of the time-averaged autocorrelation is equal to the ensamble autocorrelation.

A random process $X(t)$ is ***ergodic*** if time-averages are the same for all sample functions, and are equal to the corresponding ensamble averages.

In an ergodic process, all its statistics can be obtained from a single sample function.

A stationary process $X(t)$ is called ***ergodic in the mean*** if $\langle x(t)\rangle = \bar{X}$,

and ***ergodic in the autocorrelation*** if $\langle x(t)x(t+\tau)\rangle = R_{XX}(\tau)$.

Correlations and Power Spectral Densities

Assume all random processes WSS:

I. Autocorrelation $R_{XX}(\tau)$:

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]\,.$$

Properties of $R_{XX}(\tau)$: $R_{XX}(-\tau) = R_{XX}(\tau)$, $|R_{XX}(\tau)| \le R_{XX}(0)$, $R_{XX}(0) = E[X^2(t)]$.

II. Cross-Correlation $R_{XY}(\tau)$: $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$.

Properties of $R_{XY}(\tau)$: $R_{XY}(-\tau) = R_{XY}(\tau)$, $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)R_{YY}(0)}$, $|R_{XY}(\tau)| \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$.

III. Autocovariance $C_{XX}(\tau)$: $C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2$

IV. Cross-Covariance $C_{XY}(\tau)$: $C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\bar{Y}$.

Two processes are ***(mutually) orthogonal*** if $R_{XY}(\tau) = 0$, and ***uncorrelated*** if $C_{XY}(\tau) = 0$.

V. Power Spectrum Density $S_{XX}(\omega)$:

The ***power spectral density*** $S_{XX}(\omega)$ is the ***Fourier transform*** of $R_{XX}(\tau)$ $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau$. Thus

$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega$.

Properties: real, $S_{XX}(\omega) \ge 0$***, even fn.*** $S_{XX}(-\omega) = S_{XX}(\omega)$,

(Parseval's type relation) $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = R_{XX}(0) = E[X^2(t)]$.

VI. Cross Spectral Densities

$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau$, $S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)e^{-j\omega\tau}d\tau$.

Therefore: $R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega$, $R_{YX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)e^{j\omega\tau}d\omega$.

Since $R_{XY}(\tau) = R_{YX}(-\tau)$, then $S_{XY}(\omega) = S_{YX}(-\omega) = S^*_{YX}(\omega)$.

Random Processes in Linear Systems

I. System Response:

LTI system with impulse response $h(t)$, and output $Y(t) = L[X(t)] = h(t) * X(t) = \int_{-\infty}^{\infty} h(\zeta)X(t-\zeta)d\zeta$.

II. Mean and Autocorrelation of Output:

$$E[Y(t)] = \bar{Y}(t) = E\left[\int_{-\infty}^{\infty} h(\zeta)X(t-\zeta)d\zeta\right] = \int_{-\infty}^{\infty} h(\zeta)E[(X(t-\zeta)]d\zeta = \int_{-\infty}^{\infty} h(\zeta)\bar{X}(t-\zeta)d\zeta = h(t) * \bar{X}(t).$$

$$R_{YY}(t_1, t_2) = E[Y(t_1)Y(t_2)] = E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\zeta)h(\mu)X(t_1-\zeta)X(t_2-\mu)(d\zeta)d\mu\right] =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\zeta)h(\mu)E[X(t_1-\zeta)X(t_2-\mu)]d\zeta d\mu = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\zeta)h(\mu)R_{XX}(t_1-\zeta, t_2-\mu)d\zeta d\mu$$

If input is WSS then $E[Y(t)] = \bar{Y} = \int_{-\infty}^{\infty} h(\zeta)\bar{X}d\zeta = \bar{X}\int_{-\infty}^{\infty} h(\zeta)d\zeta = \bar{X}H(0)$, the mean of the output is a cons.

$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\zeta)h(\mu)R_{XX}(t_2-t_1+\zeta-\mu)d\zeta d\mu$, $R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\zeta)h(\mu)R_{XX}(\tau+\zeta-\mu)d\zeta d\mu$: Y WSS

III. Power Spectral Density of Output:

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\zeta)h(\mu)R_{XX}(\tau+\zeta-\mu)e^{-j\omega\tau}d\tau d\zeta d\mu = |H(\omega)|^2 S_{XX}(\omega),$$

$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_{XX}(\omega)e^{j\omega\tau}d\omega$. Average power of output is:

$$E[Y^2(t)] = R_{YY}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_{XX}(\omega)d\omega$$

Special Classes of Random Processes

I. Gaussian Random Process:

II. White Noise:

A random process $X(t)$ is *white noise* if $S_{XX}(\omega) = \frac{\eta}{2}$. Its autocorrelation is $R_{XX}(\tau) = \frac{\eta}{2}\delta(\tau)$.

III. Band-Limited White Noise:

A random process $X(t)$ is *band-limited white noise* if $S_{XX}(\omega) = \begin{cases} \frac{\eta}{2}, & |\omega| \le \omega_B \\ 0, & |\omega| > \omega_B \end{cases}$.

Then $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\omega_B}^{\omega_B} \frac{\eta}{2} e^{j\omega\tau} d\omega = \frac{\eta\omega_B}{2\pi}\frac{\sin\omega_B\tau}{\omega_B\tau}$.

IV. Narrowband Random Process:

Let $X(t)$ be WSS process with zero mean. Let its power spectral density $S_{XX}(\omega)$ be nonzero only in a narrow frequency band of width $2W$ which is very small compared to a center frequency ω_c, then we have a *narrowband random process*. When white or broadband noise is passed through a narrowband linear filter, narrowband noise results. When a sample function of the output is viewed on oscilloscope, the observed waveform appears as a sinusoid of random amplitude and phase. Narrowband noise is conveniently represented by $X(t) = V(t)\cos[\omega_c t + \phi(t)]$, where $V(t)$ and $\phi(t)$ are the *envelop function* and *phase function*, respectively. From trigonometric identity of the cosine of a sum we get the *quadrature representation* of process:

$$X(t) = V(t)\cos\phi(t)\cos\omega_c t - V(t)\sin\phi(t)\sin\omega_c t = X_c(t)\cos\omega_c t - X_s(t)\sin\omega_c t$$

where

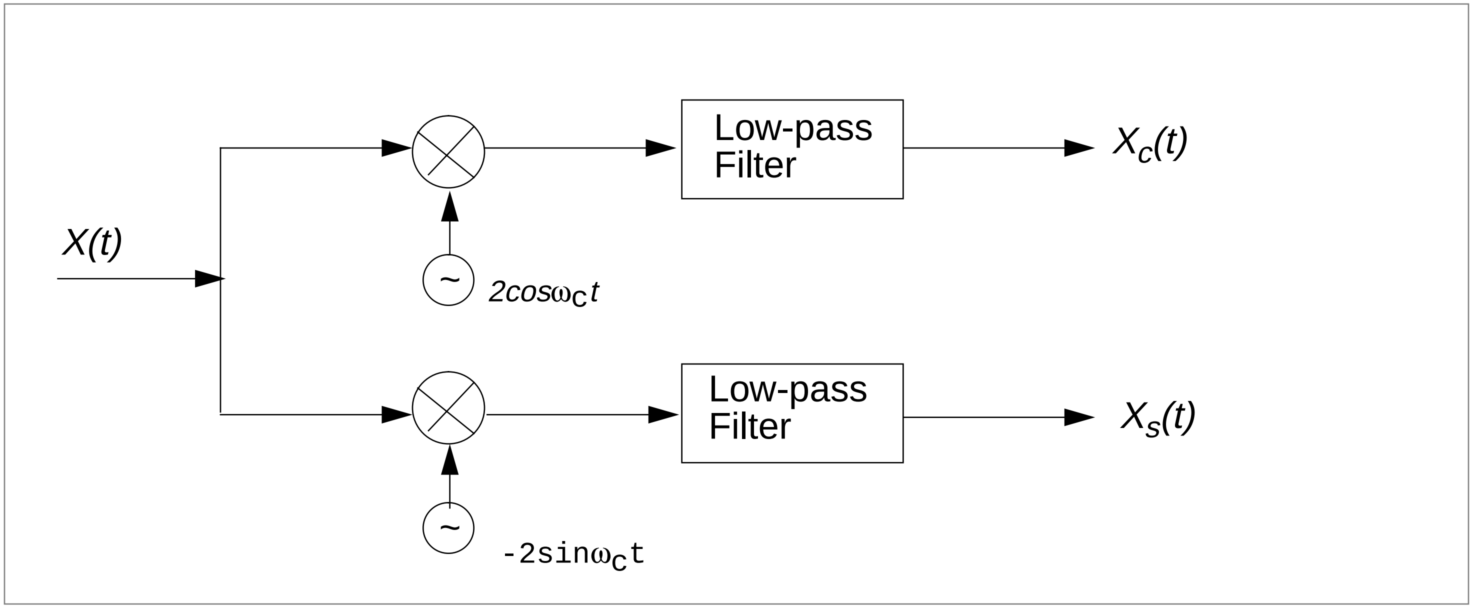
$$X_c(t) = V(t)\cos\phi(t), \quad in-phase\ component$$

$$X_s(t) = V(t)\sin\phi(t), \quad qudrature\ component$$

$$V(t) = \sqrt{X_c^2(t) + X_s^2(t)}$$

$$\phi(t) = \operatorname{atan}\frac{X_s(t)}{X_c(t)}$$

To detect the quadrature component $X_c(t)$, and the in-phase component $X_s(t)$ from $X(t)$ we use:



Properties of $X_c(t)$ ***and*** $X_s(t)$ ***:***

1- Same power spectrum:

$$S_{X_c}(\omega) = S_{X_s}(\omega) = \begin{cases} S_{XX}(\omega-\omega_c) + S_{XX}(\omega+\omega_c), & |\omega| \le W \\ 0, & else \end{cases}$$

2- Same mean and variance as $X(t)$:

$$\bar{X}_c = \bar{X}_s = \bar{X} = 0, \; and \;\; \sigma^2_{X_c} = \sigma^2_{X_s} = \sigma^2_X$$

3- $X_c(t)$ and $X_s(t)$ are uncorrelated: $E[X_c(t)X_s(t)] = 0$

4- If input process is gaussian, then so are the in-phase and quadrature components.

5- If $X(t)$ is gaussian, then for a fixed t, $V(t)$ is a random variable with Rayleigh distribution, and $\phi(t)$ is a random variable uniformly distributed over $[0, 2\pi]$.