Random Variables

Definition:

A random variable, X, is a real-valued function defined on a sample space S. It is a mapping from S into $\Re$.

Examples:

Ex. 1: Toss coin. $S = \{H, T\}$. Define X such that $X(H) = 1, \quad X(T) = 0$.

Ex. 2: Toss coin until head comes up. $S = \{H, (T, H), (T, T, H), \ldots, (T, T, \ldots, T, H)\}$.
Define $X(H) = 1, \quad X(T, H) = 2, \quad X(T, T, H) = 3, \ldots$.

Ex. 3: Toss a coin 4 times for each experiment: $S = \{(H, H, H, H), (H, H, H, T), \ldots, (T, T, T, T)\}$. Define X to be number of heads in sample point, e.g. $X(H, H, T, H) = 3$.

Ex. 4: Toss two dice: $S = \{(n, m):1 \le n, m \le 6\}$, define $X((n, m)) = n + m$.

Events Defined From Random Variables:

Let X be a R.V. on S, $S = \{s_1, s_2, \ldots\}$, defined by $X(s) = x$.

Let $\{X = 1\}$ define the event $\{s \in S\text{:X}(s) = 1\}$, similarly
let $\{X = a\} = \{s \in S\text{:X}(s) = a\}$, and
$\{a \le X < b\} = \{s \in S\text{:a} \le X(s) < b\}$.

For Ex. 1, $\{X = 1\}=\{H\}$, therefore $p_X(1) = 1/2$. $\{X = 0\}=\{T\}$, therefore $p_X(0) = 1/2$. $\{X = 2\}=\varnothing$, $\{X = \sqrt{3}\}=\varnothing$, $\{0 \le X < 1\}=S$.

For Ex. 2, $\{X = 1\} = \{H\}$, $\{X = 5\} = \{T, T, T, T, H\}$, $\{X < 3\} = \{(H), (T, H)\}$.
$p_X(1) = 1/2$, $p_X(2) = 1/4$,..., $p_X(N) = 1/2^N$.

For Ex. 3, $\{X = 3\} = \{(H, H, H, T), (H, H, T, H), (H, T, H, H), (T, H, H, H)\}$, $\{X = 0\} = \{T, T, T, T\}$.
$p_X(0) = 1/16$, $p_X(1) = 1/4$, $p_X(2) = 6/16 = 3/8$, $p_X(3) = 1/4$, $p_X(4) = 1/16$. For $p_X(2)$, the number of sample points with two heads is 4x3/2!=6, we have total of 2^4 sample points of equal probability, hence the 6/16 answer.

For Ex. 4, $S = \{X = 2\} + \{X = 3\} + \ldots + \{X = 12\}$, a partition. Then $P(S) = p_X(2) + p_X(3) + \ldots + p_X(12) = 1$.

Definition: *A random variable that has only discrete experimental values (finite or countably infinite) is called a* ***Discrete Random Variable****.*

Probability Mass Function (Probability Distribution):

Let X be a discrete R.V., and x be an experimental value. Define $p_X(x) = P\{X{=}\mathrm{x}\}$ as the ***probability mass function.***
A general property of probability mass function *(pmf)*: $\sum_{all\ \mathrm{x}} p_X(x) = 1$.

Bernoulli Random Variable:

Experiment: "Bernoulli Trial" $S = \{\text{Success}, \text{Failure}\} = \{s, f\}$. $P(\{s\}) = p$, $P(\{f\}) = 1 - p$.

Bernoulli R.V. $X(\{\text{Success}\}) = 1$, $X(\{\text{Failure}\}) = 0$. $p_X(1) = p$, $p_X(0) = 1 - p$.

Binomial R.V. (Repeated Trials):

Experiment: N independent Bernoulli trials. $S = \{(s, s, \ldots, s), (s, s, \ldots, s, f), \ldots, (f, f, \ldots, f)\}$. 2^N sample points. Define the ***Binomial R. V.*** by mapping each sample point into an integer (subset of reals) equal to the number of successes. How many points are there with n successes and N-n failures? N!/n!(N-n)! Therefore $p_X(n) = \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} = \binom{N}{n} p^n (1-p)^{N-n}$, where $\binom{N}{n} \equiv C(N, n) = \frac{N!}{(N-n)!n!}$.

Binomial Theorem: $(a+b)^N = \sum_{n=0}^{N} \binom{N}{n} a^n b^{N-n}$.

Binomial R.V. X: $p_X(n) = \binom{N}{n} p^n (1-p)^{N-n}, n = 0, 1, \ldots, N$.

Then $\sum_{n=0}^{N} p_X(n) = \sum_{n=0}^{N} \binom{N}{n} p^n (1-p)^{N-n} = (p + (1-p))^N = 1$.

$\binom{N}{n}$ = number of ways of having n successes and $N - n$ failures.

Then $\sum_{n=0}^{N} \binom{N}{n}$ = total number of sample points $= (1+1)^N = 2^N$.

(Cumulative) Distribution Function

I. Discrete R. V.

Discrete R. V. X. pmf: $p_X(x_i) = P(\{X = x_i\})$

Define CDF $F_X(x) = P(\{X \le x\}) = \sum_{x_i \le X} p_X(x_i)$

Properties of $F_X(x)$*:*

1- $\lim_{x \to \infty} F_X(x) = 1$

2- $\lim_{x \to -\infty} F_X(x) = 0$

3- $P(\{a < X \le b\}) = F_X(b) - F_X(a)$, i.e. $F_X(x)$ is a nondecreasing function.

II. Continuous R. V.

Define $F_X(x) = P(\{X \le x\})$, and $f_X(x) = \frac{d}{dx}F_X(x)$ = ***Probability Density Function.***

$$P(\{a < X \le b\}) = F_X(b) - F_X(a) = \int_a^b f_X(x)\, dx.$$

Properties of $f_X(x)$*:*

1- $f_X(x) \ge 0$

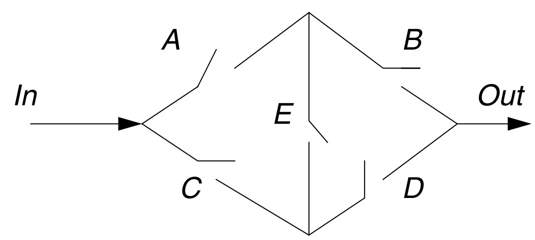
2- $\int_{-\infty}^{\infty} f_X(x) = 1$

Examples

5.1 In the switching network shown, the switches operate independently. Each switch closes with probability p, and remains open with probability $1 - p$.

a- Find the probability that a signal at the input will be rec

b- Find the conditional probability that switch E is open, g



5.2 In a certain Village 20% of the population has disease *D*. A test is administered which has the property that if a person has *D*, the test will be positive 90% of the time, and if he does not have *D*, the test will be positive 30% of the time. All those whose test is positive are given a drug which invariably cures the disease, but produces a characteristic rash 25% of the time. Given that a person picked at random has the rash, what is the probability that he actually had *D* to begin with?