Formulation of the General Problem

The general form of an equation whose roots can be studied by the root locus method is

$$D(s) + KN(s) = 0 \tag{8.1-11}$$

where K is the parameter to be varied. For now, we take the functions $D(s)$ and $N(s)$ to be polynomials in s with constant coefficients, and we consider the case $K \geq 0$.

Another standard form of the problem is obtained by rewriting (8.1-11) as

$$1 + KP(s) = 0 \tag{8.1-12}$$

where

$$P(s) = \frac{N(s)}{D(s)} \tag{8.1-13}$$

Plotting Guides for the Primary Root Locus

Standard Form:

$$1 + KP(s) = 0, \qquad K \geq 0$$
$$P(s) = N(s)/D(s)$$
$$N(s) = s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0$$
$$D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$
$$m \leq n$$

Terminology:

The *zeros* of $P(s)$ are the roots of $N(s) = 0$.
The *poles* of $P(s)$ are the roots of $D(s) = 0$.

Guide 1

The root locus plot is symmetric about the real axis. This is because complex roots occur in conjugate pairs. Thus, we need deal with only the upper half-plane of the plot.

Guide 2

The number of loci equals the number of poles of P(s).

Guide 3

The loci start at the poles of P(s) *with* K = 0 *and terminate with* K = ∞ *either at the zeros of* P(s) *or at infinity.*

Guide 4

The root locus can exist on the real axis only to the left of an odd number of real poles and/or zeros; furthermore, it must exist there.

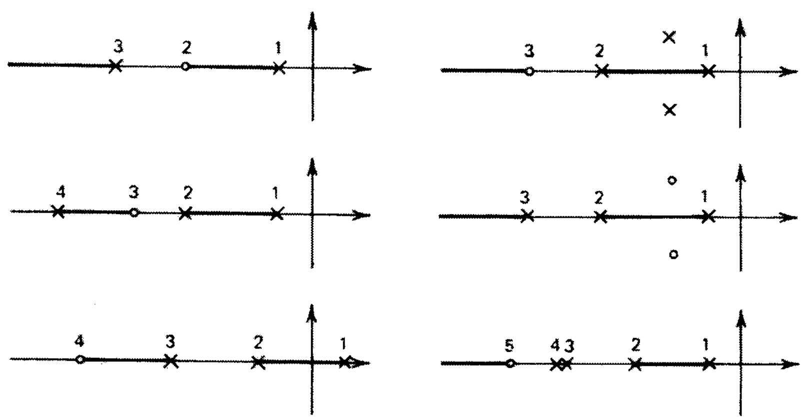


FIGURE 8.8 Location of the root locus on the real axis for various pole-zero configurations. The locus off the real axis is not shown.

Guide 5

The locations of breakaway and break-in points are found by determining where the parameter K attains a local maximum or minimum on the real axis.

Example 8.1

Determine the root locus plot for the equation

$$1 + \frac{K(s+6)}{s(s+4)} = 0 \qquad \textbf{(8.2-3)}$$

The poles are $s = 0, -4$; the zero is $s = -6$. Thus, there are two loci. One starts at $s = 0$ and the other at $s = -4$, with $K = 0$. One path terminates at $s = -6$, while the other must terminate at $s = \infty$. This information is given by Guides 2 and 3.

Guide 4 shows that the locus exists on the real axis between $s = 0$ and $s = -4$ and to the left of $s = -6$. Therefore, the two paths must break away from the real axis between $s = 0$ and $s = -4$. From the location of the termination point at $s = -6$, we know that the locus must return to the real axis. From Guide 1, we see that both paths must break in at the same point.

These points are found from Guide 5 as follows. Solve (8.2-3) for K and compute dK/ds.

$$K = -\frac{s(s+4)}{s+6}$$

$$\frac{dK}{ds} = -\frac{(s+6)(2s+4) - s(s+4)}{(s+6)^2} = 0$$

This is satisfied for finite s if the numerator is zero.

$$s^2 + 12s + 24 = 0$$

The candidates are

$$s = -6 \pm 2\sqrt{3} = -2.54, \ -9.46$$

There is no need to check for a minimum or a maximum, because we know from Guide 4 that the breakaway point must be $s = -2.54$ and the break-in point $s = -9.46$. This leaves only the shape of the locus off the real axis to be determined.

Elementary geometry can be used to show that the root locus of the equation

$$(s+b)(s+c) + K(s+a) = 0 \qquad \textbf{(8.2-4)}$$

is a circle centered on the zero at $s = -a$ if $a, b, c > 0$ and $a > b > c$. This case is shown in Figure 8.10. The radius of the circle can be determined once the breakaway and break-in points are found.

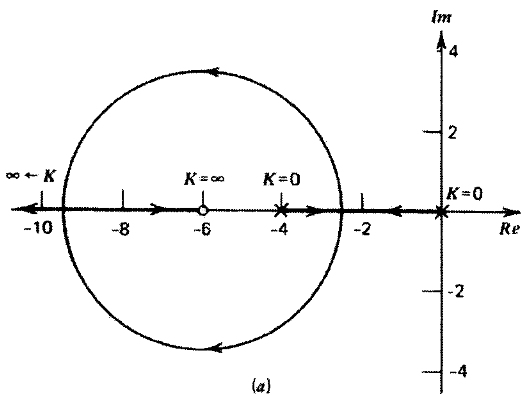


FIGURE 8.9 **(a) Root-locus plot for $s(s+4) + K(s+6) = 0$, for $K \geq 0$.**

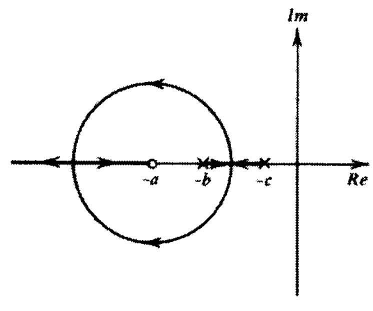


FIGURE 8.10 **Root-locus plot for $(s+b)(s+c) + K(s+a) = 0$ for $K \geq 0$ and $a > b > c > 0$.**

Guide 6

The loci that do not terminate at a zero approach infinity along asymptotes. The angles that the asymptotes make with the real axis are found from (8.2-5), where n *is chosen successively as* n = *+1, −1, +3, −3, ..., until enough angles have been found.*

$$\theta = \frac{n180^\circ}{Z-P} \qquad n = \pm 1, \pm 3, \ldots \tag{8.2-5}$$

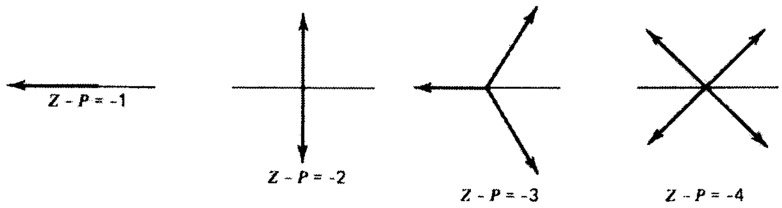


FIGURE 8.12 Asymptotic angles for commonly occurring cases.

Guide 7

The asymptotes intersect the real axis at the common point

$$\sigma = \frac{\Sigma s_p - \Sigma s_z}{P-Z} \tag{8.2-6}$$

where Σs_p *and* Σs_z *are the algebraic sums of the values of the poles and zeros.*

Guide 8

The points at which the loci cross the imaginary axis and the associated values of K *can be found by the Routh–Hurwitz criterion or by substituting* $s = i\omega$ *into the equation of interest. The frequency* ω *is the* crossover frequency.

Example 8.2

Plot the root locus for the equation

$$s^3 + 3s^2 + 2s + K = 0 \qquad \text{for } K \geqslant 0 \tag{8.2-7}$$

We can factor the equation as follows:

$$1 + \frac{K}{s(s+1)(s+2)} = 0$$

The poles are $s = 0, -1, -2$, and there are no zeros. All three paths approach asymptotes as $K \to \infty$. The locus exists on the real axis between $s = 0$ and -1 and to the left of $s = -2$.

To find the breakaway point, compute dK/ds.

$$K = -(s^3 + 3s^2 + 2s)$$

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

The candidates are $s = -0.423, -1.58$. The breakaway point must obviously be $s = -0.423$, because the locus cannot exist between $s = -1$ and $s = -2$. With this value of s, (8.2-7) gives $K = 0.385$.

The three asymptotic angles are found from (8.2-5).

$$\theta = \frac{n180^\circ}{0-3} = -n60^\circ \qquad n = \pm 1, \pm 3, \ldots$$

Thus,

$$\theta = -60^\circ, +60^\circ, \pm 180^\circ$$

Note that for the last angle it does not matter whether we use $n = +3$ or $n = -3$. The intersection is found from (8.2-6).

$$\sigma = \frac{0 + (-1) + (-2) - 0}{3 - 0} = -1$$

The two paths that start at $s = 0$ and -1 approach the $\pm 60^\circ$ asymptotes. The path starting at $s = -2$ follows the 180° asymptote and thus lies entirely on the real axis.

The 60° asymptotes indicate that two paths will cross the imaginary axis and generate two unstable roots. Substituting $s = i\omega$ into (8.2-7) gives

$$-i\omega^3 - 3\omega^2 + 2i\omega + K = 0$$

or

$$i\omega(2 - \omega^2) = 0$$
$$K = 3\omega^2$$

The solution $\omega = 0$, $K = 0$ corresponds to the pole at $s = 0$. The solution of interest is $\omega = \pm\sqrt{2}$, $K = 6$. The locus can now be sketched. This is shown in Figure 8.13*a*. The system has three roots. All are real and negative for $0 \leqslant K \leqslant 0.385$. For $0.385 < K < 6$, the system is stable with two complex roots and one real root. For $K > 6$, the system is unstable due to two complex roots with positive real parts.

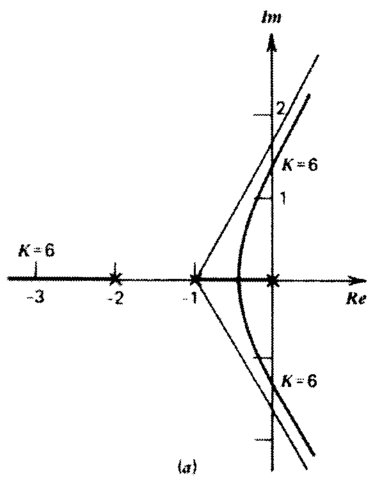
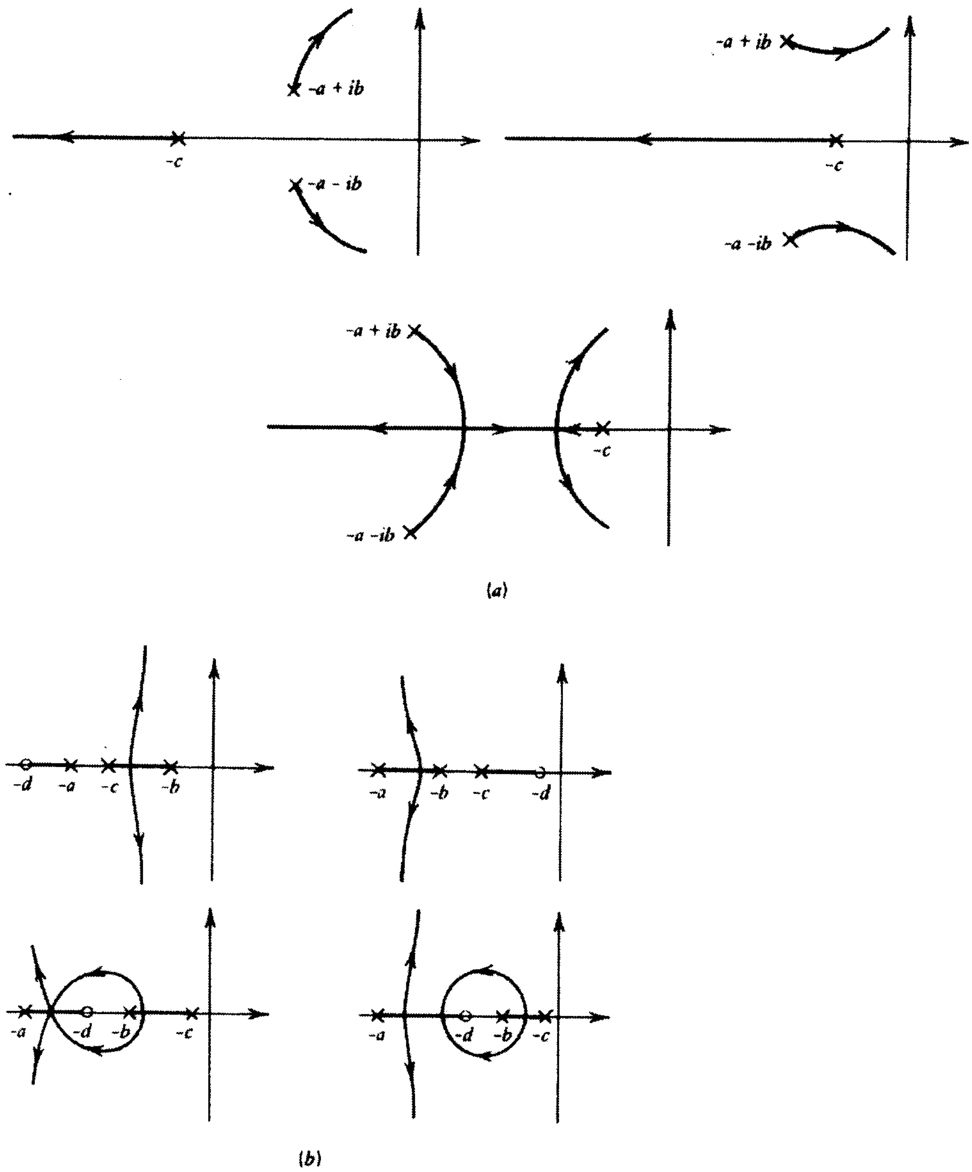


FIGURE 8.13 **(*a*) Root-locus plot for $s(s+1)(s+2) + K = 0$, for $K \geqslant 0$.**

%%
MATLAB
%%

1. We have to have our characteristic equation (the denominator of our transfer function) and rearrange it as den(s)+K*num(s); i.e., separate the polynomials with K from without K.
2. REMEMBER that the "den" and "num" are NOT the denominator and numerator of our actually system. They are for drawing use only here.
3. For example, if our C.E. is as (s^3+2*s^2-5*s)+K*(s^2+5*s+4)=0, then we type "den=[1 2 -5 0]" and "num=[1 5 4]".
4. Type "sys=tf(num,den)" to define the sys for draw root locus plot purpose.
5. Type "rlocus(sys)"



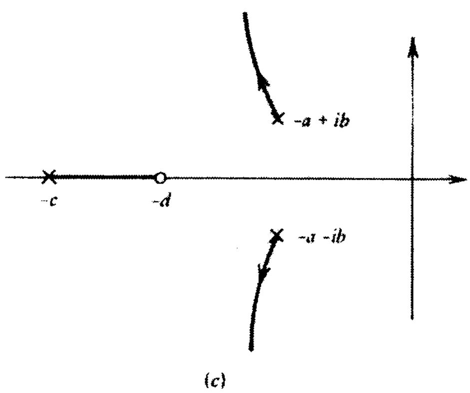


FIGURE 8.15 Some common root-locus forms not seen in the chapter examples. (*a*) $(s+c)(s+a+ib)(s+a-ib)+K=0,\ K\geqslant 0$. (*b*) $(s+a)(s+b)(s+c)+K(s+d)=0,\ K\geqslant 0$ (see also Figures 8.16 and 9.12*c*). (*c*) $(s+c)(s+a+ib)(s+a-ib)+K(s+d)=0,\ K\geqslant 0$.