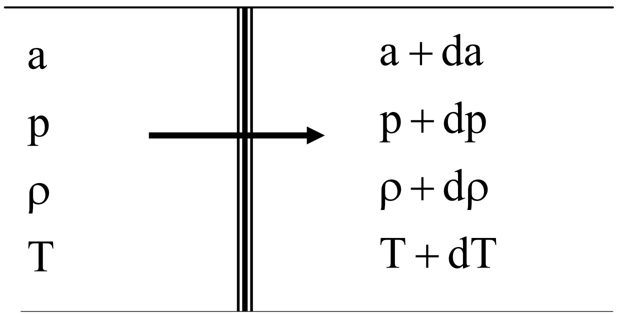
SPEED OF SOUND, a AS A NONDIMENSIONALIZING PARAMETER

1 Dimensional, isentropic flow



Momentum Equation

$$F = pA$$

$$F = m\,a = m\frac{dV}{dt} = p\,A$$

$$A\big(p - (p + dp)\big) = \rho\,A\,a\big((a - da) - a\big)$$

$$A\,dp = \rho\,A\,a\,da$$

$$\frac{dp}{a} = \rho\,da \quad \text{Euler Equation for steady motion}$$

Continuity Equation

$$\text{mass}_1 = \text{mass}_2$$

$$\rho\,A\,a = A\,(\rho + d\rho)(a - da)$$

$$\rho\,a = \rho\,a + \rho\,da + a\,d\rho - d\rho\,da$$

$$\rho\,da = a\,d\rho \qquad 0$$

combining momentum and continuity,

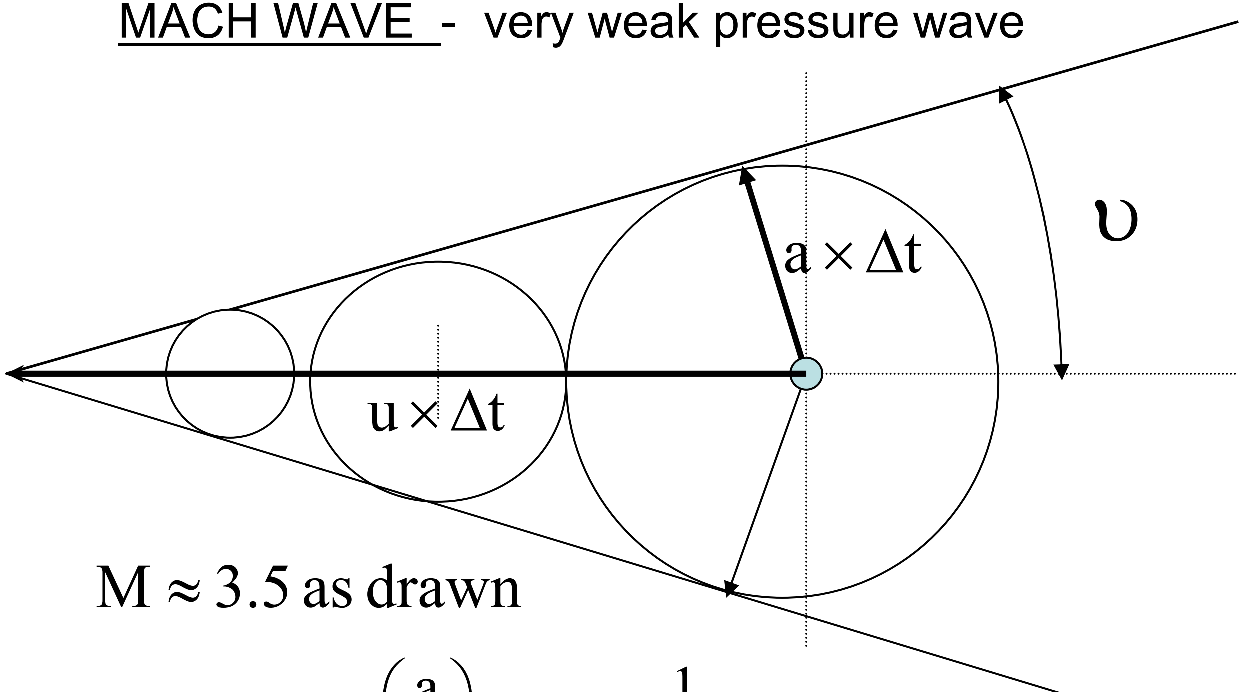
$$a^2 = \left(\frac{dp}{d\rho}\right)_{s=conctant}$$

general result

not limited to an ideal gas

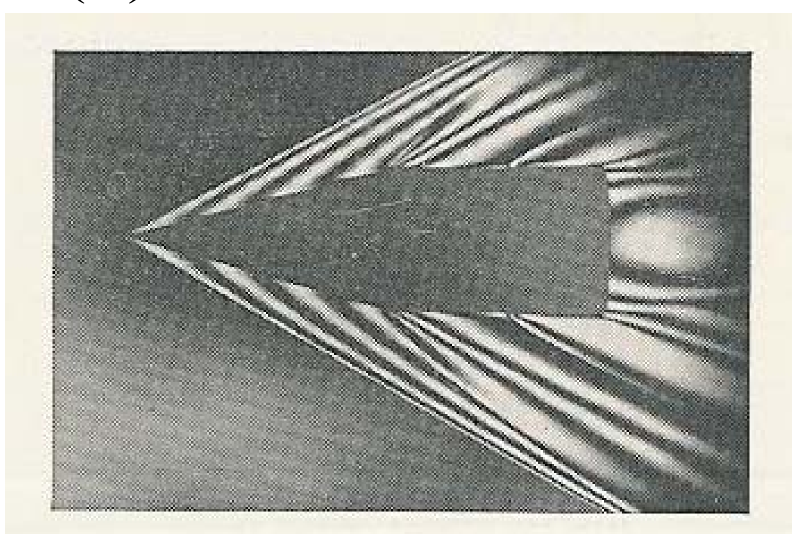
$$\text{Mach Number,} \quad M = \frac{u}{a}$$

<u>MACH WAVE</u> - very weak pressure wave



$M \approx 3.5$ as drawn

$$\text{Mach Angle, } \upsilon = \text{Sin}^{-1}\left(\frac{a}{u}\right) = \text{Sin}^{-1}\frac{1}{M} \quad (4.1)$$



Interferometer image

SPEED OF SOUND IN AN IDEAL GAS

for an isentropic process

$$p\,v^{\gamma} = \text{constant} = \frac{p}{\rho^{\gamma}}$$

$$\ln p - \gamma \ln \rho = \text{constant}$$

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho}$$

$$\left(\frac{dp}{d\rho}\right) = \gamma \frac{p}{\rho}$$

for an ideal gas

$$\frac{p}{\rho} = R\,T$$

$$\left(\frac{dp}{d\rho}\right)_{s=\text{constant}} = \gamma\,R\,T = a^2$$

air at 500 R

$$a = \sqrt{\gamma\,g\,R\,T}$$

$$a = \sqrt{1.4 \times \frac{1545.15\ \text{ft lbf/lbmole R}}{28.97\ \text{lbm/ lbmole}} \times 32\,\frac{\text{lbm ft}}{\text{lbf sec}^2} \times 500\ \text{R}}$$

$$a = 1096.4\ \text{ft/sec}$$

helium at 300 K

$$a = a = \sqrt{\gamma R T \times 1000}$$

$$a = \sqrt{1.67 \times \frac{8.314\ \text{kJ/kgmole K}}{4\ \text{kg/kgmole}} \times 1000\,\frac{\text{kgm/sec}^2}{\text{kJ}} \times 300}$$

$$a = 1020.5\ \text{m/sec}$$

STAGNATION PROPERTIES AS NONDIMENSIONALIZING PARAMETERS

properties after a gas is brought to rest isentropically, s=constant
properties with u=0, M=0

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_1^2}{2} \quad \text{Energy Equation}$$

for $u_1 = 0$

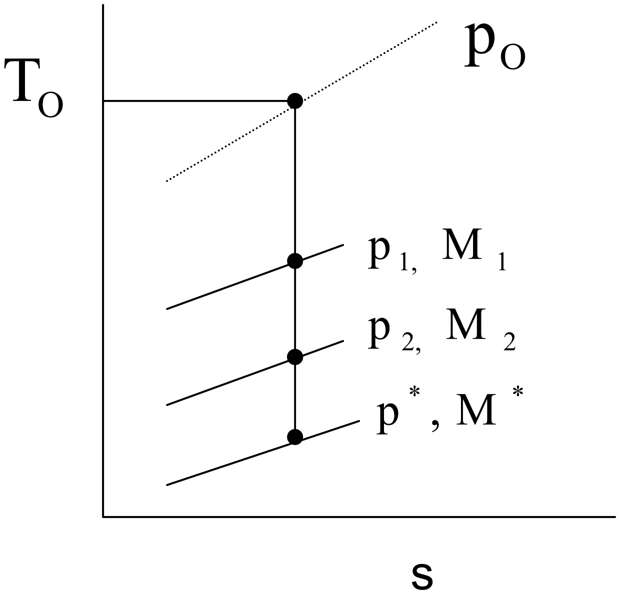
$$c_p T_O = c_p T + \frac{u^2}{2}$$

$$T_O = T + \frac{u^2}{2c_p}$$

$$\frac{T_O}{T}, \ \frac{p_O}{p}, \ \frac{\rho_O}{\rho}$$

since the stagnation process is isentropic entropy = constant

$$\frac{s_O}{s} = 1, \quad s = \text{constant}$$



1-D ISENTROPIC FLOW OF AN IDEAL GAS

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$ energy equation (only the energy equaton is required)

for $u_1 = 0$

$$h_O = h_2 + \frac{u_2^2}{2}$$

$$c_p T_O = c_p T + \frac{u^2}{2} \qquad [h = c_p T]$$

$$\frac{T_O}{T} = 1 + \frac{u^2}{2 c_p T}$$

since $c_p = \frac{\gamma R}{\gamma - 1}$ $\quad [c_p = \text{constant}]$

$$\frac{T_O}{T} = 1 + \frac{(\gamma - 1)\, u^2}{2\, \gamma\, R\, T}$$

$$\frac{T_O}{T} = 1 + \frac{\gamma - 1}{2} M^2 \quad \text{Table C.10, (12.25)}$$

$$\left[a = \sqrt{\gamma R T}, \quad M = u/a\right]$$

$$\left[p v^\gamma = \text{constant} \Leftrightarrow \Delta s = 0\right]$$

since for isentropic flow, $p v^\gamma = \text{constant}$

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma - 1} \qquad (12.26)$$

$$\frac{p_O}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}} \text{Table C.10, (12.28)}$$

$$\frac{\rho_O}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma - 1}} \text{Table C.10, (12.27)}$$

TABLE C.10 Isentropic Flow Functions (One-Dimensional, Ideal Gas, $\gamma = 1.4$)

M	T/T_0	p/p_0	ρ/ρ_0
0.00	1.0000	1.0000	1.0000
0.02	0.9999	0.9997	0.9998
0.04	0.9997	0.9989	0.9992
0.06	0.9993	0.9975	0.9982
0.08	0.9987	0.9955	0.9968
0.10	0.9980	0.9930	0.9950
0.12	0.9971	0.9900	0.9928
0.14	0.9961	0.9864	0.9903
0.16	0.9949	0.9823	0.9873
0.18	0.9936	0.9777	0.9840
0.20	0.9921	0.9725	0.9803
0.22	0.9904	0.9669	0.9762
0.24	0.9886	0.9607	0.9718
0.26	0.9867	0.9541	0.9670
0.28	0.9846	0.9470	0.9619
0.30	0.9823	0.9395	0.9564

REAL GAS AND IDEAL GAS PROPERTY MODELS

$$h_O = h_1 + \frac{u^2}{2} \quad \text{Energy Equation}$$

Ideal Gas	Real Gas	Property Module
		T, p, ρ, s, h, u
$M = u / a$	$M = u / a$	
$a = \sqrt{\gamma RT} = f(1\ \text{property})$	$a = \left(\frac{\partial p}{\partial \rho}\right)_{s=\text{constant}}$	$a = f(\text{any 2 properties})$
$h = c_p T = f(1\ \text{property})$	$h = \int c_p(T)\,dT$	$h = f(\text{any 2 properties})$
isentropic process $\Delta s = 0$	$\Delta s = 0$	
$pv^\gamma = \text{constant}$		
$\Delta s = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$	$s = \int (du + pv)\,dT$	$s = f(\text{any 2 properties})$
$c_p = \frac{R\gamma}{\gamma - 1}$		

Example

At a point in a 1 – D isentropic flow, $p = 5 psi, T = 530\ R$ and $p_O = 150$.

What is the Mach Number, and stagnation temperture ?

Since the flow is isentropic, p_O = constant (T_O is also constant)

$$\frac{p_O}{p} = \frac{150}{5} = 30 = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$M^2 = \left(30^{\frac{\gamma-1}{\gamma}} - 1\right) / \left(\frac{\gamma - 1}{2}\right)$$

$$M = 2.866$$

$$\frac{T_O}{T} = \left(1 + \frac{\gamma - 1}{2} M^2\right) = 30^{.2857} = 2.642$$

$$T_O = 530 \times 2.642 = 1400\ R$$

Table C.10 Solution (table uses $\frac{p}{p_O}$, text equaton is $\frac{p_O}{p}$)

$$\frac{p}{p_O} = \frac{5}{150} = .0333$$

Table C.10 @ , $\frac{p}{p_O} = .0333, \quad M = 2.86$

$$\frac{T_O}{T} = 2.624$$

Non isentropic flow :

- flow with friction
- flow with heat transfer
- flow across shocks

SONIC STATE AS A NONDIMENSIONALIZING PARAMETER

M has the disadvantages of being:

- a function of varables other than velocity
- becoming large at large Mach numbers

The properties where the Mach number is 1, designated*, the sonic flow condition, can be used as a nondimensionalizing parameter.

Characteristic Mach Number

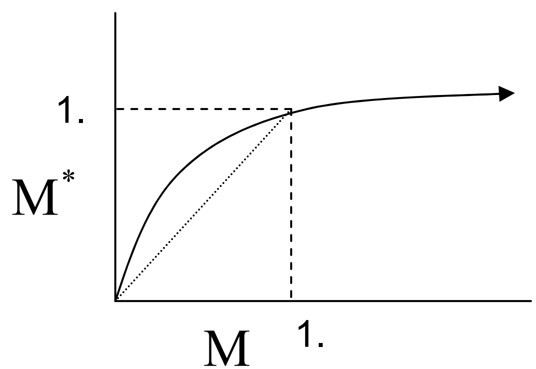
$$M^{*2} = \frac{u^2}{a^{*2}} = \frac{u^2}{u^{*2}} = M^2 \frac{a^2}{a^{*2}}$$

$$M = 0, \quad M^* = 0$$

$$M < 0, \quad M^* < 0$$

$$M > 0, \quad M^* > 0$$

$$M = \infty, \quad M^* = \frac{\gamma+1}{\gamma-1}$$



energy equation

$$h_1 + u_1^2/2 = h_2 + u_2^2/2$$

$$c_p T_1 + u_1^2/2 = c_p T_2 + u_2^2/2$$

since, $c_p = \gamma R/(\gamma - 1)$

$$\frac{\gamma R T_1}{\gamma-1} + \frac{u_1^2}{2} = \frac{\gamma R T_2}{\gamma-1} + \frac{u_2^2}{2}$$

$$\frac{a_1^2}{\gamma-1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma-1} + \frac{u_2^2}{2}$$

at $M = 1, \quad a = u = a^*$

$$\frac{a_1^2}{\gamma-1} + \frac{u_1^2}{2} = \frac{a_2^{*2}}{\gamma-1} + \frac{a_2^{*2}}{2}$$

dividing by u^2

$$\frac{(a^2/u^2)}{\gamma-1} + \frac{1}{2} = \frac{\gamma+1}{2(\gamma-1)} \frac{a^{*2}}{u^2}$$

$$\frac{(1/M^2)}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)} \frac{a^{*2}}{u^2} - \frac{1}{2}\frac{(\gamma-1)}{(\gamma-1)}$$

$$\frac{1}{M^2} = \frac{\gamma+1}{2M^{*2}} - \frac{(\gamma-1)}{2}$$

$$M^2 = \frac{2}{\dfrac{\gamma+1}{2M^{*2}} - \dfrac{(\gamma-1)}{2}} \qquad (3.37)$$

1-D ISENTROPIC FLOW OF AN IDEAL GAS

Using sonic flow rather than sonic velocity as a dimensionless reference,

$T^*, p^*, \rho^*, a^*, u^*$ are defined as the properties where $M = 1$.

using, $\dfrac{T_O}{T} = 1 + \dfrac{\gamma - 1}{2} M^2$ with $T = T^*$,

$$\frac{T^*}{T_O} = \frac{1}{1 + \frac{\gamma - 1}{2}} = \frac{2}{\gamma + 1}$$

$$\frac{a^{*2}}{a_O^2} = \frac{\gamma R T^*}{\gamma R T_O} = \frac{2}{\gamma + 1}$$

$$\frac{p^*}{p_O} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma + 1}}$$

$$\frac{\rho^*}{\rho_O} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma + 1}}$$

for $\gamma = 1.4$

$$\frac{T^*}{T_O} = .833$$

$$\frac{p^*}{p_O} = .528$$

$$\frac{\rho^*}{\rho_O} = .634$$

SHOCK WAVE FORMATION

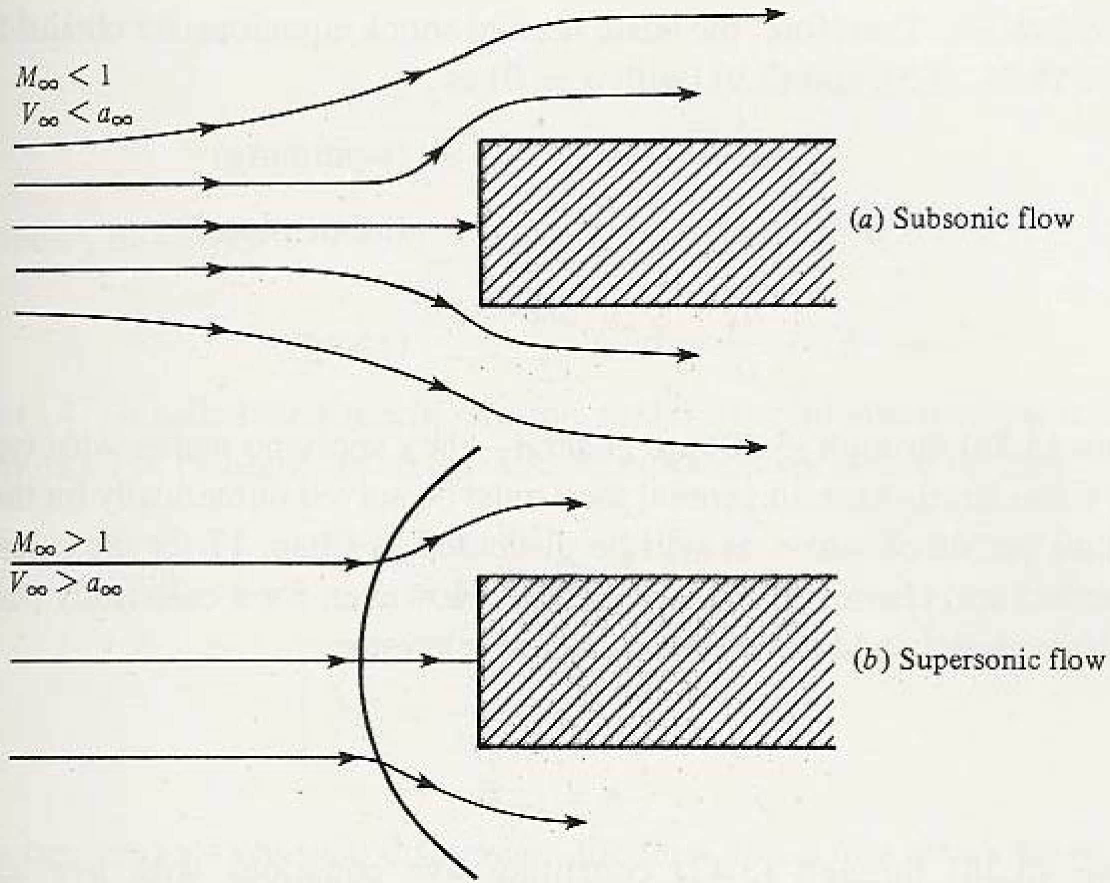
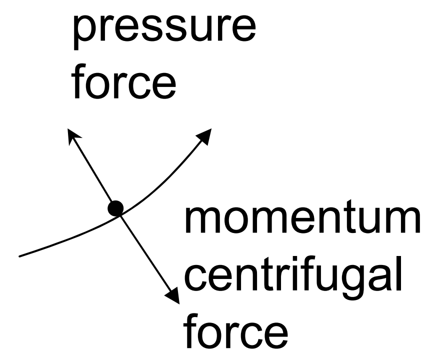
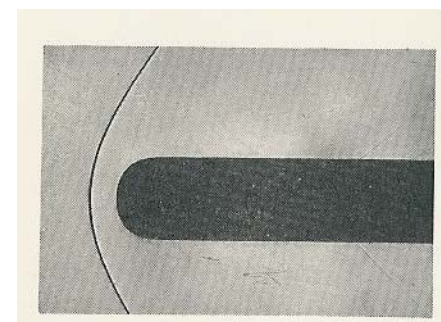


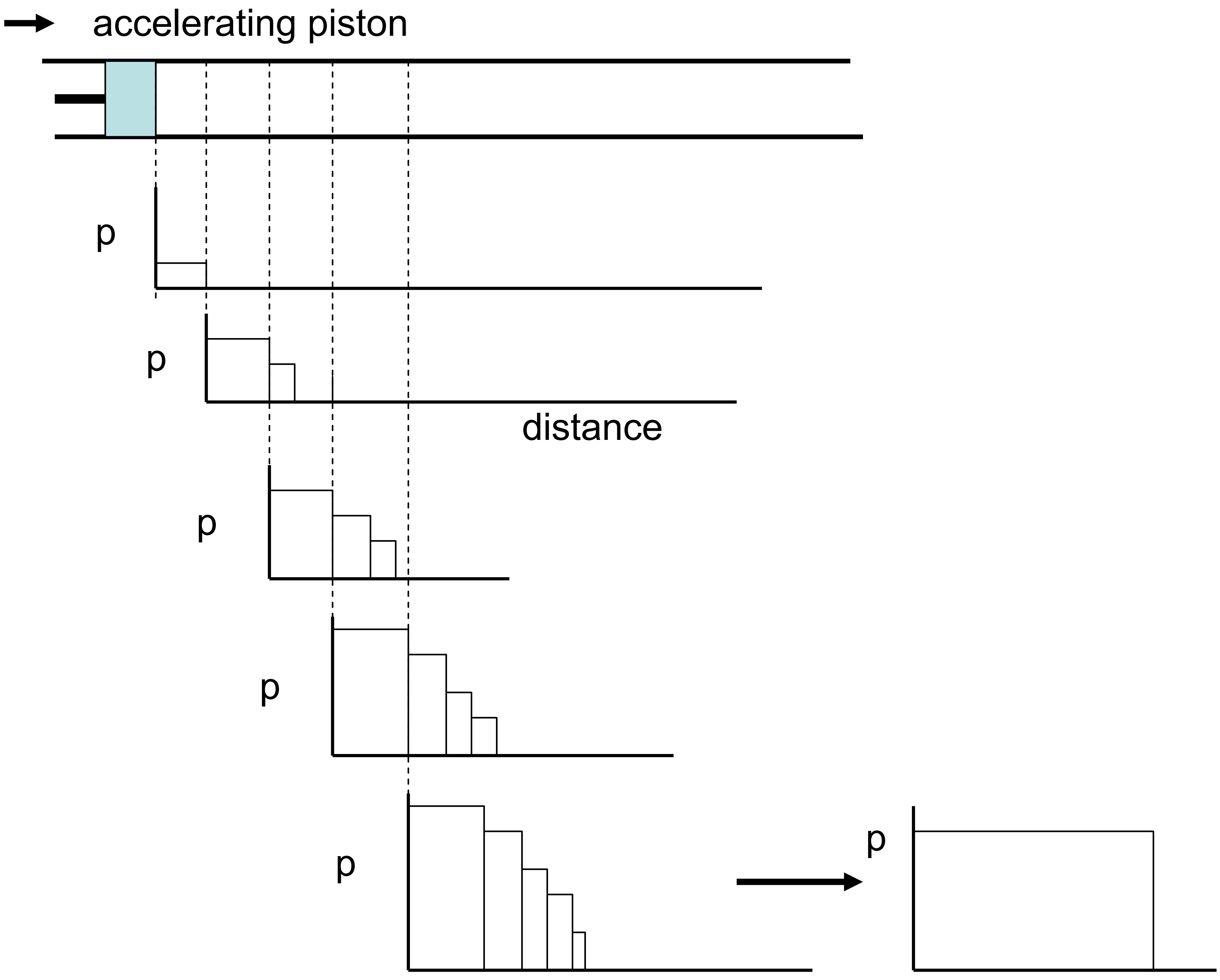
Figure 3.8 | Comparison between subsonic and supersonic streamlines for flow over a flat-faced cylinder or slab.





shadowgraph image

SHOCK WAVE FORMATION



NORMAL SHOCK

EQUATIONS

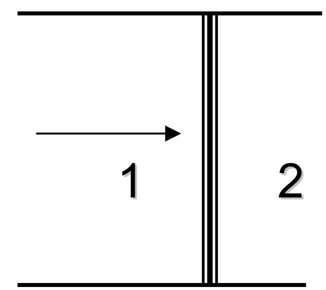
continuity equation $\rho_1 u_1 = \rho_2 u_2$

omentum equation $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$

energy equation $h_1 + \dfrac{u_1^2}{2} = h_2 + \dfrac{u_1^2}{2}$

enthalpy $h = c_p T$

sonic velocity $a^2 = \gamma RT$, ideal gas law $p = \rho RT$



$$M_2 < M_1$$
$$p_2 > p_1$$
$$T_2 > T_1$$
$$a_2 > a_1$$
$$u_2 < u_1$$
$$T_{O1} = T_{O2}$$
$$p_{O2} < p_{O1}$$

RESULTS :

$$a^{*2} = u_1 \times u_2$$

$$M_2^* = \frac{1}{M_1^*}$$

$$M_2^2 = \frac{1 + \left(\frac{(\gamma - 1)}{2} M_1^2\right)}{\gamma M_1^2 - \frac{(\gamma - 1)}{2}} \quad (12.41)\ \text{Table C.11}$$

$$\left(\frac{p_2}{p_1}\right)(12.46), \left(\frac{\rho_2}{\rho_1}\right)(12.45), \left(\frac{T_2}{T_1}\right)(12.47), \left(\frac{p_{O2}}{p_{O1}}\right)(12.28), \quad \text{Table C.11}$$

NORMAL SHOCK

sonic velocity change across a noramal shock

continuity $\rho_1 u_1 = \rho_2 u_2$

momentum $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$

energy $h_1 + \dfrac{u_1^2}{2} = h_2 + \dfrac{u_1^2}{2}$

enthalpy definition $h = c_p T$

sonic velocity $a^2 = \gamma R T$

ideal gas law $p = \rho R T$

dividing energy by continuity,

$$\frac{p_1 + \rho_1 u_1^2}{\rho_1 u_1} = \frac{p_2 + \rho_2 u_2^2}{\rho_2 u_2}$$

$$\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1$$

$$a^2 = \gamma\, R\, T = \gamma\, p/\rho \quad , p = \frac{a^2 \rho}{\gamma}$$

substituting into ,

$$\frac{a_1^2}{\gamma u_1} - \frac{a_1^2}{\gamma u_1} = u_2 - u_1$$

energy equation

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2}$$

since, $c_p = \dfrac{\gamma R}{\gamma - 1}$, and $a^2 = \gamma\, R\, T$

$$\frac{\gamma R T_1}{\gamma - 1} + \frac{u_1^2}{2} = \frac{\gamma R T_2}{\gamma - 1} + \frac{u_2^2}{2}$$

$$\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2}$$

multiplying by $(\gamma - 1)$, and

letting 2 be sonic, $u_2 = u^* = a^*, a = a^*$

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma - 1} + \frac{a^{*2}}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2}$$

$$a^2 = \frac{\gamma + 1}{2} a^{*2} - \frac{\gamma + 1}{2} u^2$$

$$a_1^2 = \frac{\gamma + 1}{2} a_1^{*2} - \frac{\gamma + 1}{2} u_1^2$$

$$a_2^2 = \frac{\gamma + 1}{2} a_2^{*2} - \frac{\gamma + 1}{2} u_2^2$$

NORMAL SHOCK

Mach Number change across a normal shock

$$\left(\frac{\gamma+1}{2\,\gamma u_1}a_1^{*2}-\frac{\gamma-1}{2\,\gamma}u_1\right)-\left(\frac{\gamma+1}{2\,\gamma u_2}a_2^{*2}-\frac{\gamma-1}{2\,\gamma}u_2\right)=u_2-u_1$$

multiplying 1 terms by $\frac{u_1}{u_1}$ and 2 terms by $\frac{u_2}{u_2}$

$$\frac{\gamma+1}{2\,\gamma u_1\,u_2}(u_2-u_1)a^{*2}+\frac{\gamma-1}{2\gamma}(u_2-u_1)=u_2-u_1$$

$$\frac{\gamma+1}{2\,\gamma u_1\,u_2}a^{*2}+\frac{\gamma-1}{2\,\gamma}=1$$

$$a^{*2}+\frac{\gamma-1}{2\,\gamma}\frac{2\gamma u_1 u_2}{\gamma+1}=\frac{2\,\gamma u_1 u_2}{\gamma+1}$$

$$a^{*2}=\frac{2\,\gamma u_1 u_2}{\gamma+1}\left(1-\frac{\gamma-1}{2\,\gamma}\right)$$

$$a^{*2}=\frac{2\gamma u_1 u_2}{\gamma+1}\left(\frac{2\gamma-\gamma-1}{2\,\gamma}\right)=u_1 u_2$$

$$a^{*}=u_1 u_2$$

$$1=\frac{u_1}{a^*}\frac{u_2}{a^*}=M_1^*\,M_2^*$$

$$M_2^*=\frac{1}{M_1^*}$$

substituting

$$M^{*2}=\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}\quad(3.37)$$

for M_1^* and M_2^*

$$M_2^2=\frac{1+\left(\frac{(\gamma-1)}{2}M_1^2\right)}{\gamma\,M_1^2-\frac{(\gamma-1)}{2}}$$

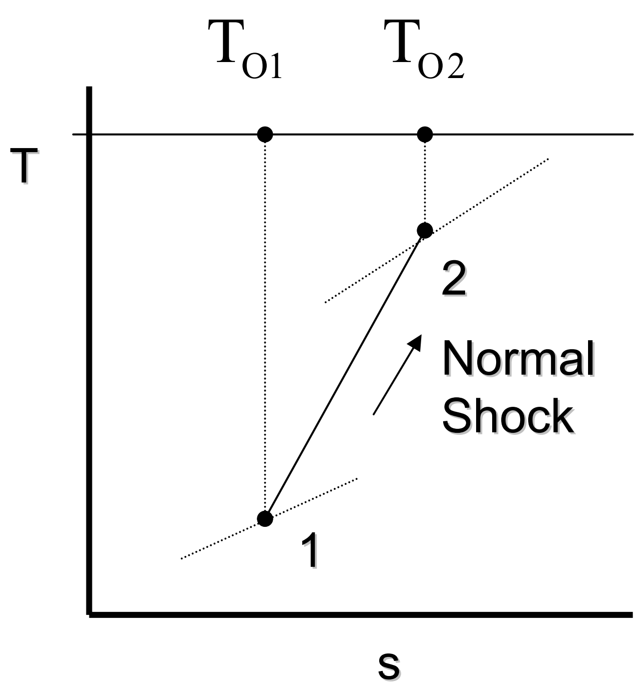
(12.41) or Table C.11

ENTROPY CHANGE ACROSS A NORMAL SHOCK

a shock is adiabatic

$$q = 0 = c_p\left(T_{O2} - T_{O1}\right)$$

$$T_{O2} = T_{O1} = \text{constant}$$



$$\Delta s = s_2 - s_1 = s_{02} - s_{O1}$$

$$\Delta s = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$

$$\Delta s = c_p \ln\left(\frac{T_{O2}}{T_{O1}}\right) - R \ln\left(\frac{p_{O2}}{p_{O1}}\right)$$

$$T_O = \text{constant}$$

$$\Delta s = -R \ln\left(\frac{p_{O2}}{p_{O1}}\right)$$

although $p_2 > p_0$

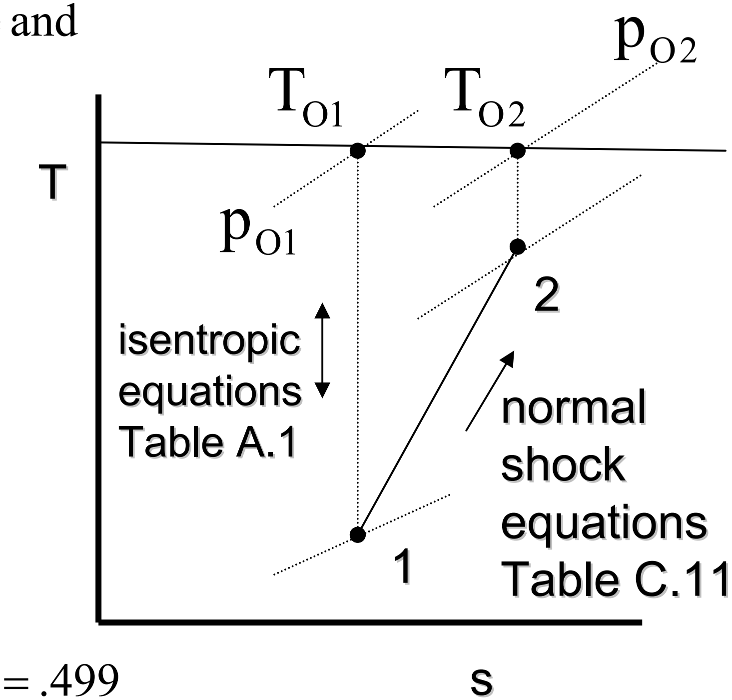
$$p_{O2} < p_{01} \Rightarrow \Delta s = +$$

TABLE C.11 Normal Shock Flow Functions (One-Dimensional, Ideal Gas, $\gamma = 1.4$)

M_1	M_2	p_{0_2}/p_{0_1}	T_2/T_1	p_2/p_1	ρ_2/ρ_1
1.00	1.000	1.000	1.000	1.000	1.000
1.02	0.9805	1.000	1.013	1.047	1.033
1.04	0.9620	0.9999	1.026	1.095	1.067
1.06	0.9444	0.9998	1.039	1.144	1.101
1.08	0.9277	0.9994	1.052	1.194	1.135
1.10	0.9118	0.9989	1.065	1.245	1.169
1.12	0.8966	0.9982	1.078	1.297	1.203
1.14	0.8820	0.9973	1.090	1.350	1.238
1.16	0.8682	0.9961	1.103	1.403	1.272
1.18	0.8549	0.9946	1.115	1.458	1.307
1.20	0.8422	0.9928	1.128	1.513	1.342
1.22	0.8300	0.9907	1.141	1.570	1.376
1.24	0.8183	0.9884	1.153	1.627	1.411
1.26	0.8071	0.9857	1.166	1.686	1.446
1.28	0.7963	0.9827	1.178	1.745	1.481
1.30	0.7860	0.9794	1.191	1.805	1.516

Example

Before a normal shock in a duct, $M = 2.5$, $p_O = 100$ psi, $T_O = 800$ R.
Find T, P and M after the shock. What is the entropy change and stagnation pressure change across the shock?



before the shock, Table C.10, Isentropic Flow

$$@\ M_1 = 2.5,\ \text{Table C.10},\quad \frac{p_1}{p_O} = .0585,\quad \frac{T_1}{T_O} = .4444$$

$$p_1 = 100 \times .0585 = 5.85\ \text{psi},\qquad T_1 = 800 \times .4444 = 355.5\ \text{R}$$

after the shock, Table C.11, Normal Shock

$$@\ M_1 = 2.5,\ \text{TableC.10},\quad \frac{p_2}{p_1} = 7.125,\quad \frac{T_2}{T_1} = 2.137,\ \frac{p_{O2}}{p_{O1}} = .499$$

$$p_2 = 7.125 \times 5.85 = 41.68\ \text{psi},\quad T_2 = 2.137 \times 355.5 = 769.7\ \text{R},\quad p_{O2} = .499 \times 100 = 49.5\text{psi}$$

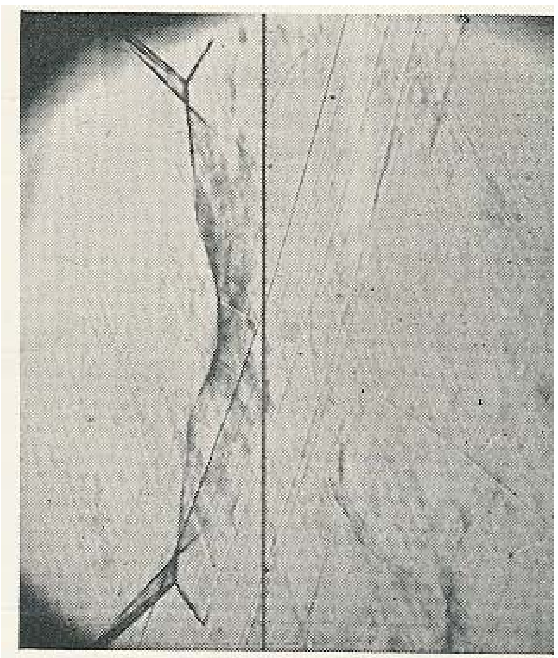
$$\Delta s = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right) \qquad \Delta s = -R \ln\left(\frac{p_{O2}}{p_{O1}}\right)$$

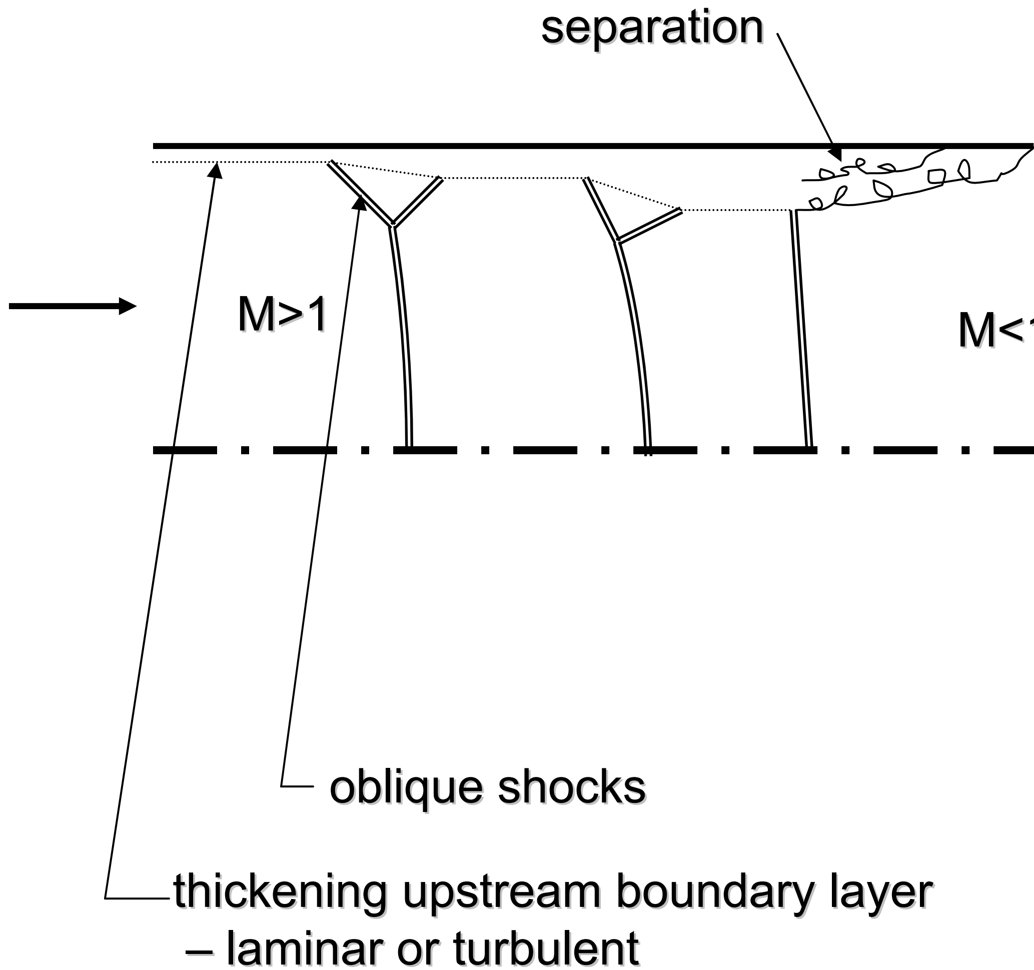
$$\Delta s = .24 \times \ln(2.137) - \frac{1545.15}{28.97 \times 778}\ln(7.125) \qquad \Delta s = -\frac{1545.15}{28.97 \times 778}\ln(.499)$$

$$\Delta s = .1822 - .1347 = +.0475\ \text{Btu/lbm R} \qquad \Delta s = .0477\ \text{Btu/lbm R}$$

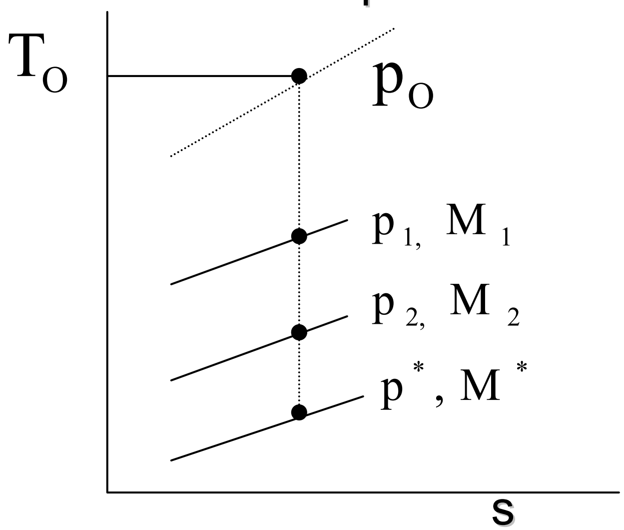
REAL SHOCK BOUNDARY LAYER INTERACTION

Actual deceleration in a duct occurs through a series of oblique and normal shocks with thickening upstream laminar or turbulent boundary layers which can separate from the duct walls before, during or after the shock process.

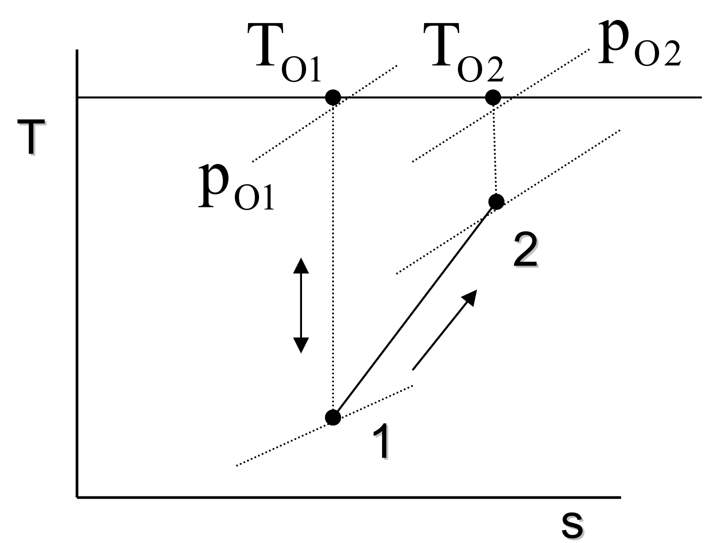




1D Isentropic



Normal Shock



SUBSONIC - SUPERSONIC FLOW WITH AREA CHANGE

Quasi 1 Dimensional, Isentropic Flow

energy equation

$$h + \frac{u^2}{2} = \text{constant}$$

differentiating, $\quad dh + u\,du = \text{constant}$

$$dh = -u\,du \qquad (1)$$

1 st and 2nd law, $\quad T\,ds = dh - \dfrac{dp}{\rho}$

$ds = 0$ since the flow is isentropic

$$dh = \frac{dp}{\rho}$$

combining with (1)

$$-u du = \frac{dp}{\rho}$$

$$du = -\frac{dp}{u\,\rho} (2)$$

pressure decreases in accelerating flow

pressure increases in decelerating flow

continuity $\rho\,A\,u = \text{constant}$

$$d(\rho\,A\,u) = 0$$

$$d(\ln \rho\,A\,u) = 0$$

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{du}{u} = 0$$

substituting (2) for du

$$\frac{d\rho}{\rho}\frac{dp}{dp} + \frac{dA}{A} - \frac{dp}{u^2\rho}$$

$$\frac{dA}{A} = \frac{dp}{\rho}\left(\frac{1}{u^2} - \frac{d\rho}{dp}\right)$$

$$\frac{dA}{A} = \frac{dp}{\rho u^2}\left(1 - \frac{u^2}{c^2}\right)$$

$$\frac{dA}{A} = \frac{(1 - M^2)}{\rho u^2} dp$$

$$\frac{dA}{dp} = A\frac{(1 - M^2)}{\rho u^2}$$

SUBSONIC - SUPERSONIC FLOW

$$\frac{dA}{dp} = A\frac{\left(1 - M^2\right)}{\rho u^2}$$

accelerating flow

nozzle

$$\frac{dp}{du} < 0$$

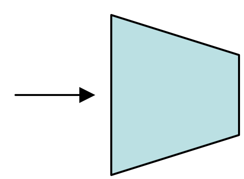
decelerating flow

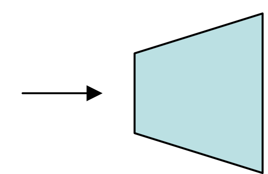
diffuser

$$\frac{dp}{du} > 0$$

SUBSONIC FLOW

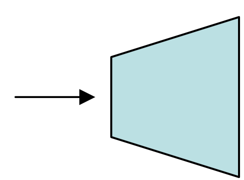
$M < 1$ $\qquad \frac{dA}{dp} > 0 \qquad \frac{dA}{du} < 0$

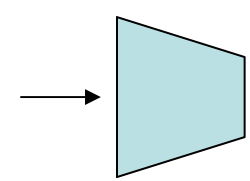




SUPERSONIC FLOW

$M > 1$ $\qquad \frac{dA}{dp} < 0 \qquad \frac{dA}{du} > 0$





General conclusion. Not limited to an ideal gas.

AREA CHANGE IN ISENTROPIC QUASI-1D FLOW OF AN IDEAL GAS

Continuity Equation $\rho^* u^* A^* = \rho\, u\, A$

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{\rho^*}{\rho_O}\frac{\rho_O}{\rho}\frac{a^*}{u} \qquad \text{(a)}$$

since the flow is isentropic

the stagnation properies are constant,

from the Energy Equation

using (12.27) at $M = 1$,

$$\frac{\rho_O}{\rho} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

$$\frac{\rho_O}{\rho^*} = \frac{\rho_O}{\rho} \text{ with } M = 1$$

$$\frac{\rho_O}{\rho^*} = \left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}}$$

$$\left(\frac{u}{a^*}\right)^2 = M^{*2} = \frac{\frac{\gamma+1}{2}M^2}{1 + \frac{\gamma-1}{2}M^2}$$

squaring (a) and subsitiuting,

$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{\rho^*}{\rho_O}\right)^2\left(\frac{\rho_O}{\rho}\right)^2\left(\frac{a^*}{u}\right)^2 = \left(\frac{\rho^*}{\rho_O}\right)^2\left(\frac{\rho_O}{\rho}\right)^2\left(\frac{1}{M^{*2}}\right)^2$$

$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}}\left(1 + \frac{\gamma+1}{2}M^2\right)^{\frac{1}{\gamma-1}}\left(\frac{1 + \frac{\gamma-1}{2}M^2}{\frac{\gamma+1}{2}M^2}\right)$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left(\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right)^{\frac{\gamma+1}{\gamma-1}} \qquad (12.33)$$

Table C.10 Isentropic Flow

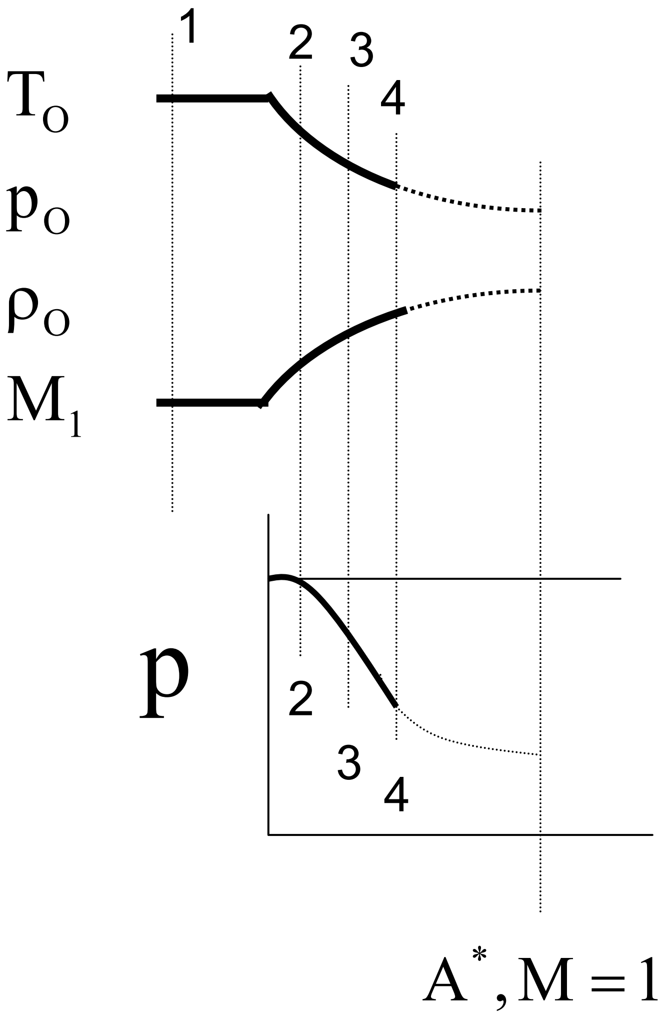
TABLE C.10 Isentropic Flow Functions (One-Dimensional, Ideal Gas, $\gamma = 1.4$)

M	T/T_0	p/p_0	ρ/ρ_0	A/A^*
0.00	1.0000	1.0000	1.0000	∞
0.02	0.9999	0.9997	0.9998	28.94
0.04	0.9997	0.9989	0.9992	14.48
0.06	0.9993	0.9975	0.9982	9.666
0.08	0.9987	0.9955	0.9968	7.262
0.10	0.9980	0.9930	0.9950	5.822
0.12	0.9971	0.9900	0.9928	4.864
0.14	0.9961	0.9864	0.9903	4.182
0.16	0.9949	0.9823	0.9873	3.673
0.18	0.9936	0.9777	0.9840	3.278
0.20	0.9921	0.9725	0.9803	2.964
0.22	0.9904	0.9669	0.9762	2.708
0.24	0.9886	0.9607	0.9718	2.496
0.26	0.9867	0.9541	0.9670	2.317
0.28	0.9846	0.9470	0.9619	2.166
0.30	0.9823	0.9395	0.9564	2.035

TABLE C.11 Normal Shock Flow Functions (One-Dimensional, Ideal Gas, $\gamma = 1.4$)

M_1	M_2	p_{0_2}/p_{0_1}	T_2/T_1	p_2/p_1	ρ_2/ρ_1
1.00	1.000	1.000	1.000	1.000	1.000
1.02	0.9805	1.000	1.013	1.047	1.033
1.04	0.9620	0.9999	1.026	1.095	1.067
1.06	0.9444	0.9998	1.039	1.144	1.101
1.08	0.9277	0.9994	1.052	1.194	1.135
1.10	0.9118	0.9989	1.065	1.245	1.169
1.12	0.8966	0.9982	1.078	1.297	1.203
1.14	0.8820	0.9973	1.090	1.350	1.238
1.16	0.8682	0.9961	1.103	1.403	1.272
1.18	0.8549	0.9946	1.115	1.458	1.307
1.20	0.8422	0.9928	1.128	1.513	1.342
1.22	0.8300	0.9907	1.141	1.570	1.376
1.24	0.8183	0.9884	1.153	1.627	1.411
1.26	0.8071	0.9857	1.166	1.686	1.446
1.28	0.7963	0.9827	1.178	1.745	1.481
1.30	0.7860	0.9794	1.191	1.805	1.516

1D Isentropic Flow
Table C.10, equations, module



$$\frac{T_O}{T} = \left(1 + \frac{\gamma - 1}{2} M^2\right) \quad (12.25)$$

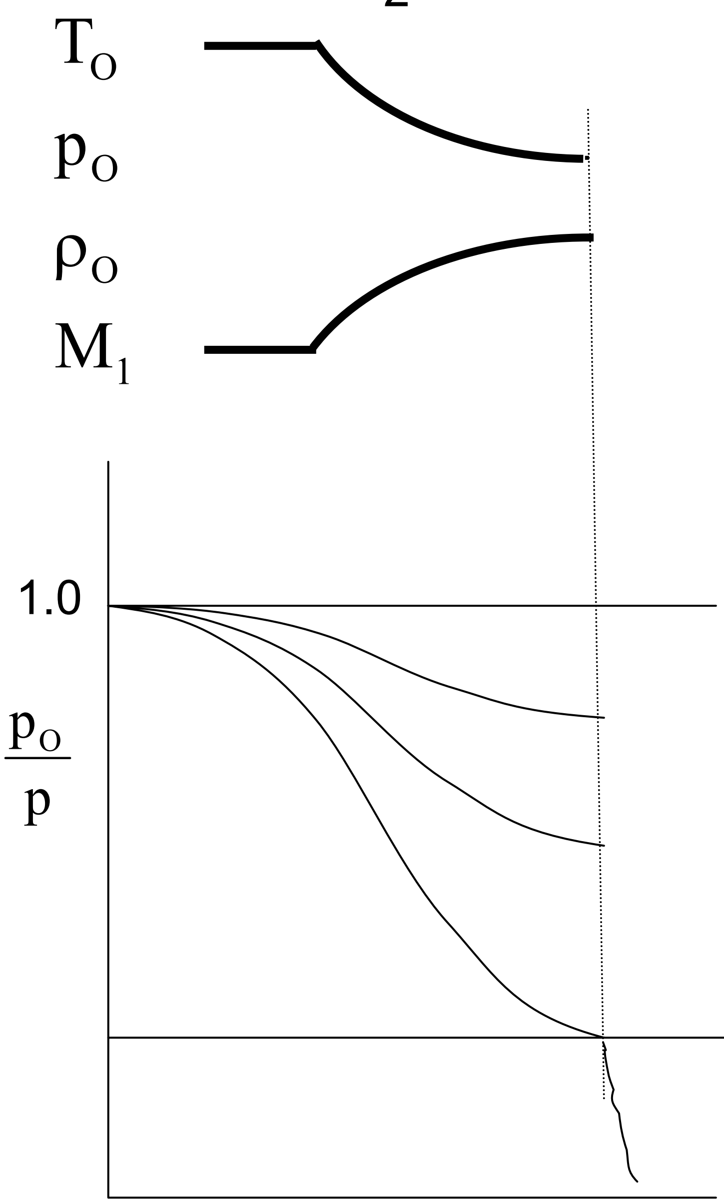
$$\frac{p_O}{p} = \left(\frac{T_O}{T}\right)^{\frac{\gamma-1}{\gamma}} \quad (12.28)$$

$$\frac{\rho_O}{\rho} = \left(\frac{T_O}{T}\right)^{\frac{1}{\gamma-1}} \quad (12.27)$$

$$\frac{A}{A^*} = \frac{1}{M^2}\left(\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2} M^2\right)\right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad (12.33)$$

Converging Nozzle

2



$$\frac{p_O}{p} < \frac{p_O}{p^*}$$

$$\frac{p_O}{p} = \frac{p_O}{p^*} = 1.893, \quad M = 1$$

$$\frac{p_O}{p} > \frac{p_O}{p^*}$$

$$M = 1$$

$$\frac{T_O}{T} = \left(1 + \frac{\gamma - 1}{2}\right) = \frac{T_O}{T^*} = 1.2$$

$$\frac{p_O}{p} = \left(\frac{T_O}{T}\right)^{\frac{\gamma-1}{\gamma}} = \frac{p_O}{p^*} = 1.893$$

$$\frac{\rho_O}{\rho} = \left(\frac{T_O}{T}\right)^{\frac{1}{\gamma-1}} = \frac{\rho_O}{\rho^*} = 1.577$$

$$\frac{A}{A^*} = 1.0$$

example

$$a_1 = \sqrt{\gamma RT} = \sqrt{1.4 \times .287 \times 350} = 375 \text{ m/sec}$$

$$M_1 = 150/375 = .4$$

isentropic @ $M_1 = .4$;

$$\frac{T_1}{T_O} = .9690, \ \frac{p_1}{p_O} = .8956, \ \frac{A}{A^*} = 1.590$$

$$\rho_1 = \frac{p}{RT} = \frac{200\text{kPa}}{.287 \times 300} = 2.32 \text{ kg/m}^3$$

$$m_1 = \rho A_1 V_1$$

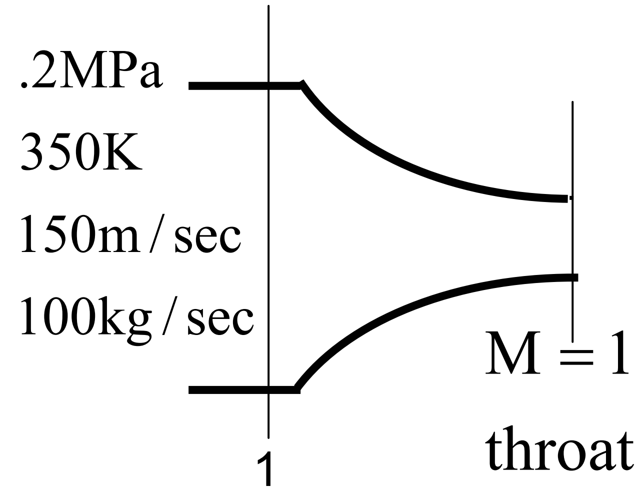
$$A_1 = \frac{m}{\rho V_1} = \frac{100}{2.32 \times 150} = .287 \text{ m}^2$$

$$A^* = A_t = A_1 \times \frac{A^*}{A_1} = .287/1.590 = .1805 \text{ m}^2$$

$$@\ M = 1, \frac{T_O}{T} = \frac{T_O}{T^*} = 1.2, \quad \frac{p_O}{p} = \frac{p_O}{p^*} = 1.893, \quad \frac{\rho_O}{\rho} = \frac{\rho_O}{\rho^*} = 1.577$$

$$T^* = \frac{T^*}{T_O} \times \frac{T_O}{T_1} \times T_1 = \frac{1}{1.20} \times \frac{1}{.9690} \times 350 = 301 \text{ K} = T_{throat}$$

$$p^* = \frac{p^*}{p_O} \times \frac{p_O}{p_1} \times p_1 = \frac{1}{1.893} \times \frac{1}{.8956} \times .2\text{MPa} = .118 \text{ MPa} = p_{throat}$$



A_t ?

T_t ?

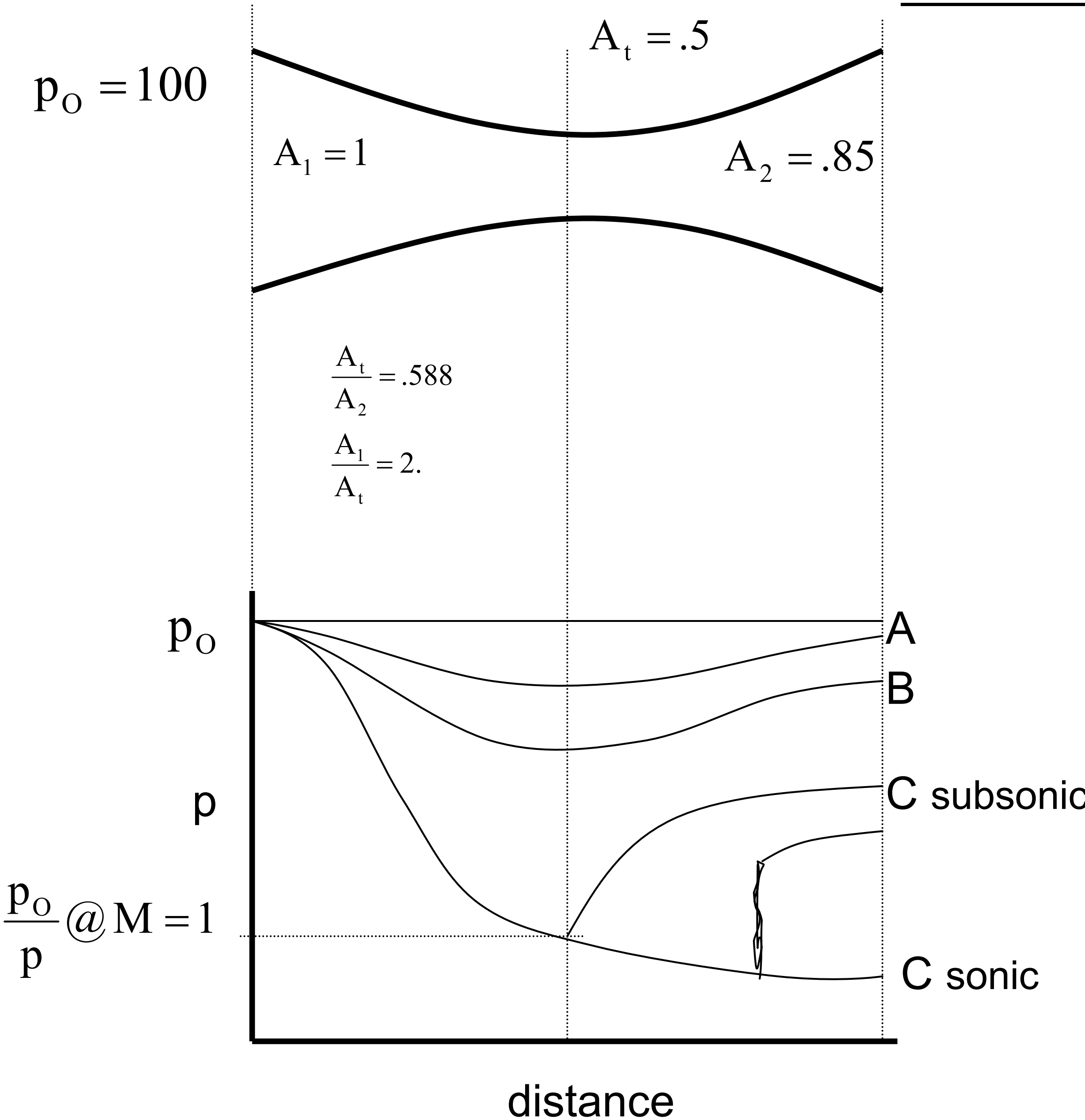
p_t ?

$$m_t = \rho_t A V_t$$

$$\rho_t = \frac{\rho^*}{\rho_O} \times \frac{\rho_O}{\rho_1} \times \rho_1$$

$$T_t = \frac{T^*}{T_O} \times \frac{T_O}{T_1} \times T_1$$

CONVERGING DIVERGING NOZZLE

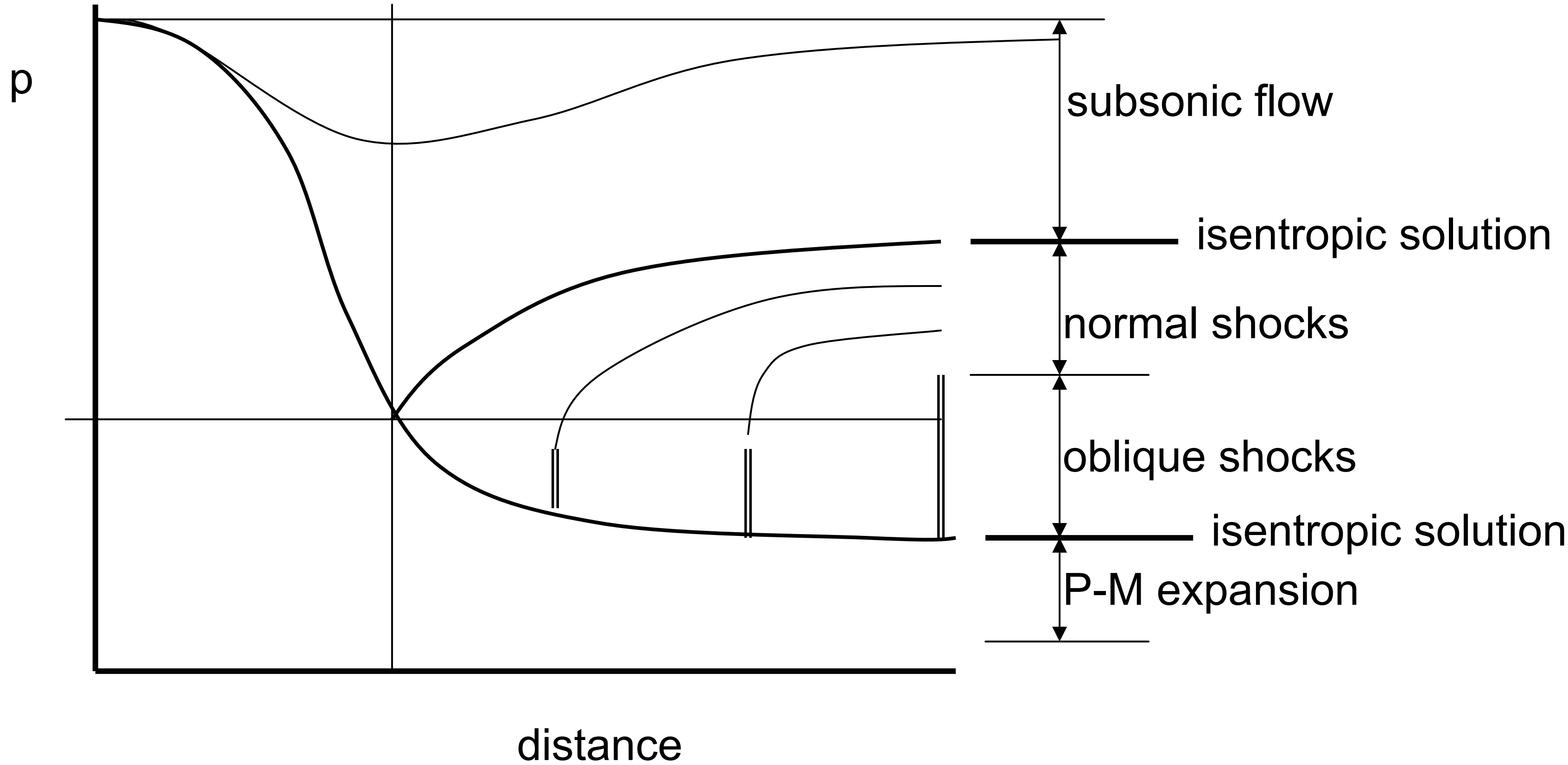


Isentropic solutions are possible at exit pressures from stagnation to the pressure at C subsonic.

Isentropic solutions are not possible at exit pressures from at C subsonic to C sonic. Exit Pressures in this range require a normal shock in the diverging Section.

An single isentropic solution is possible at exactly the pressure at C supersonic

CONVERGING-DIVERGING NOZZLE – Flow Regimes



$p_O = 10\text{atm}, M_{exit} = 3$

<u>Design exit</u>

Table C.10 $@M_{\substack{design\\exit}} = 3,$

$\frac{p_3}{p_O} = .0272$

$\left(\frac{A_e}{A^*}\right)_{supersonic} = 4.235$

$p_3 = 10 \times .0272 = .272\text{atm}$

<u>Normal exit shock @ M = 3,</u>

$M_2 = .48, \quad \frac{p_2}{p_3} = 10.33$

$p_2 = 10.33 \times .272 = 2.81\text{atm}$

<u>Throat,</u> Table C.10 @ M = 1

$\frac{p_t}{p_O} = .5283$

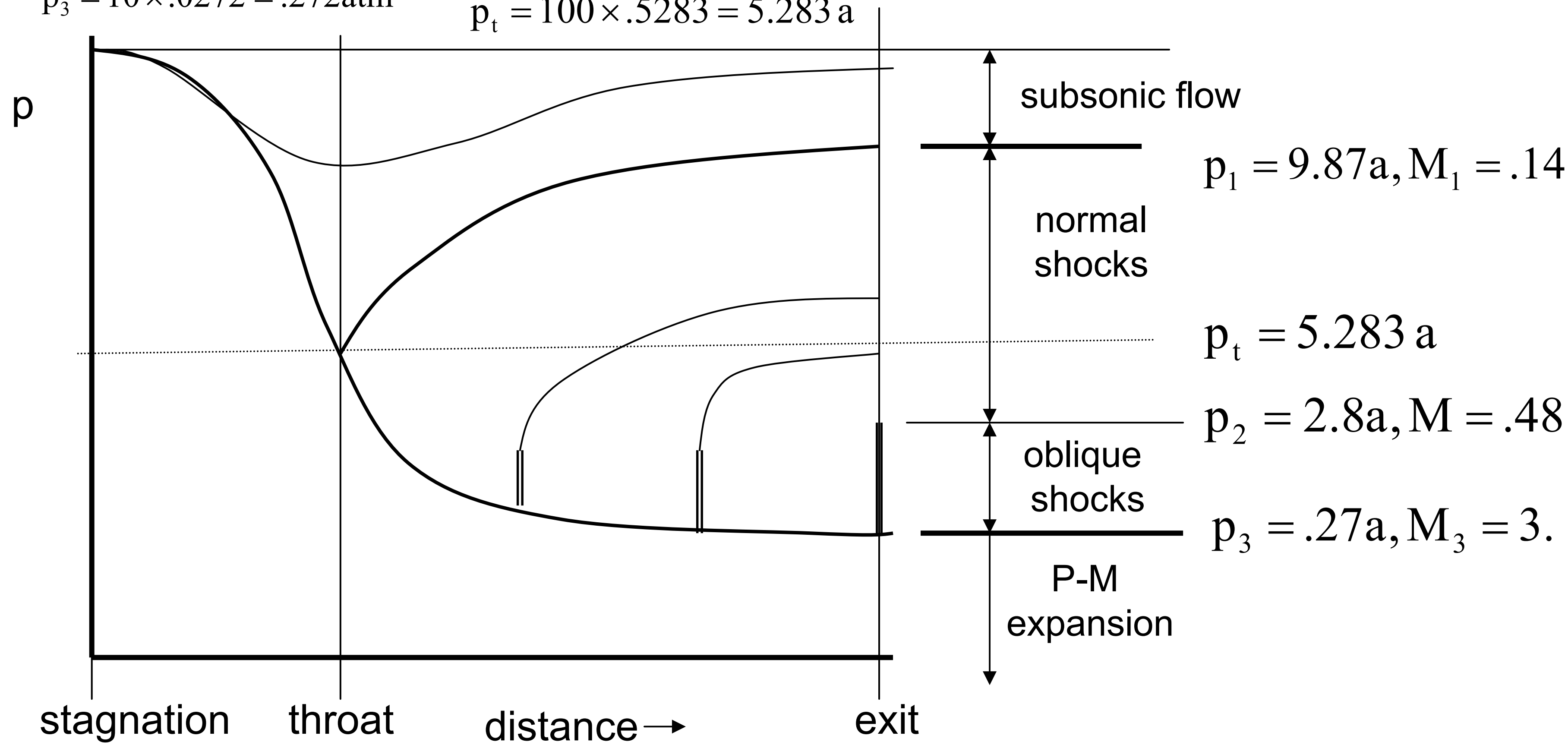
$p_t = 100 \times .5283 = 5.283\,\text{a}$

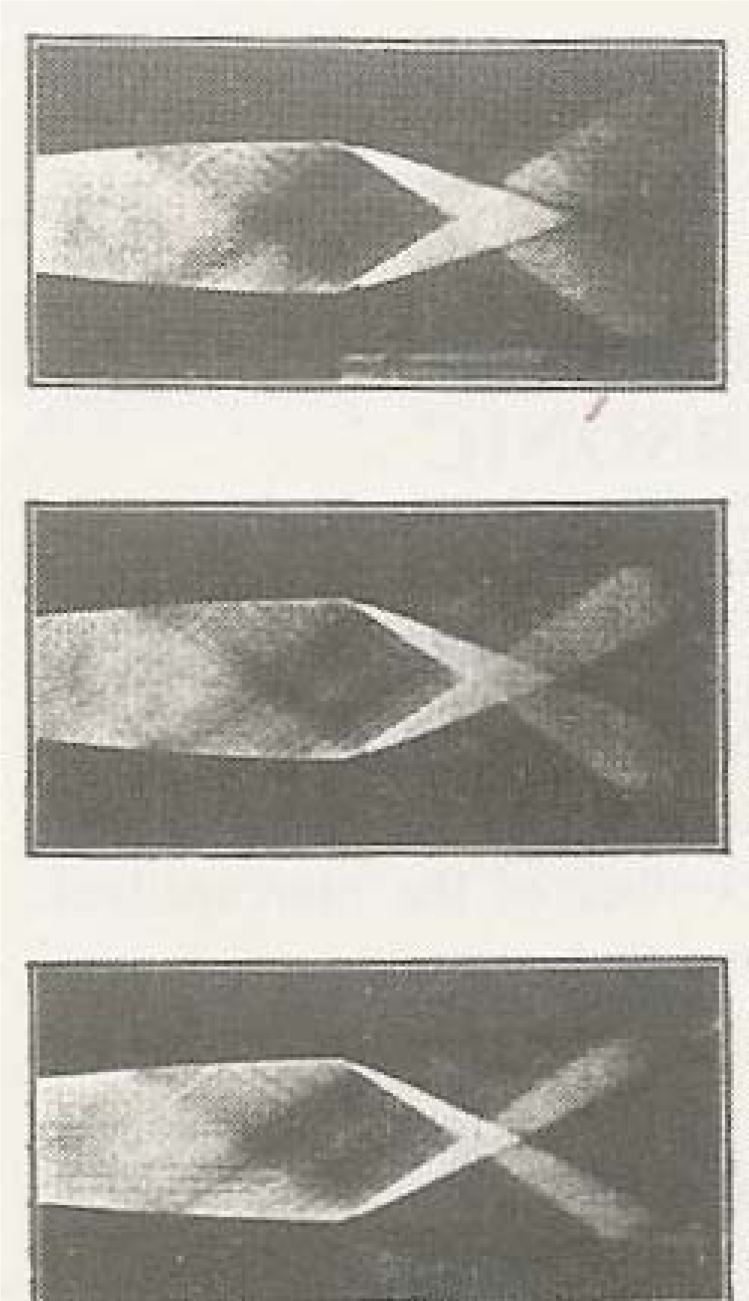
<u>subsonic exit</u>

$C.10\,@\left(\frac{A_e}{A^*}\right)_{subsonic} = 4.235,$

$\frac{p_1}{p_O} = .986, p_1 = 9.86\,\text{atm}$

$M_1 = .14$

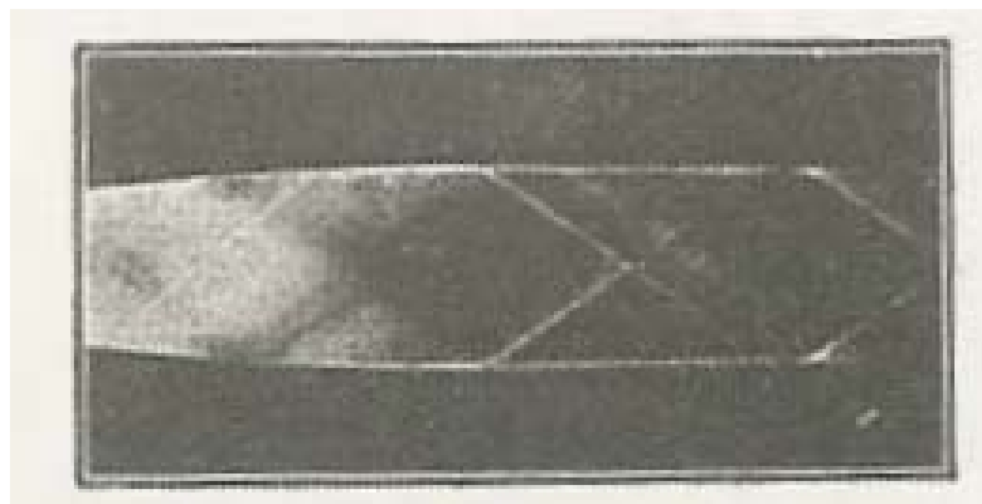




UNDER EXPANDED

$p_{exit} > p_{\substack{back \\ pressure}}$

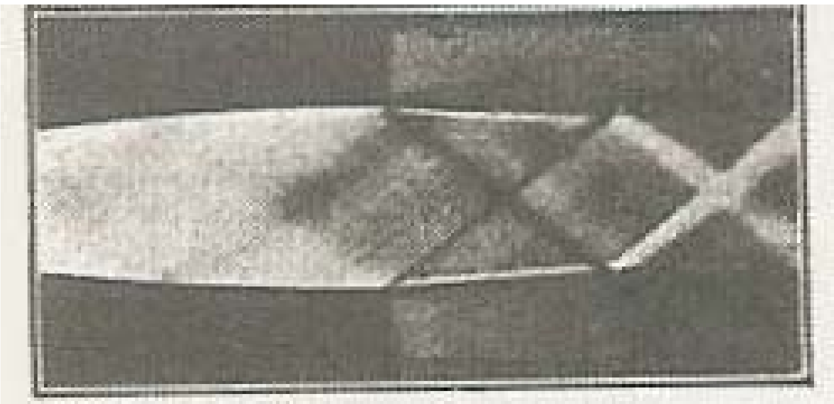
flow expands in to exit region adjustment in a Prandtl - Meyer expansion

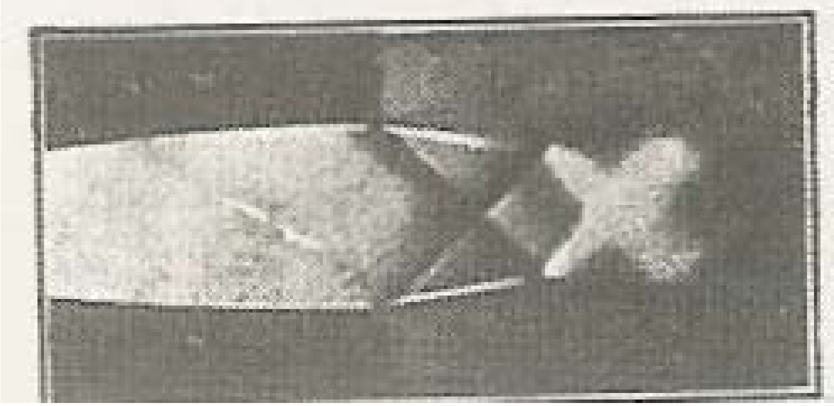


OVER EXPANDED

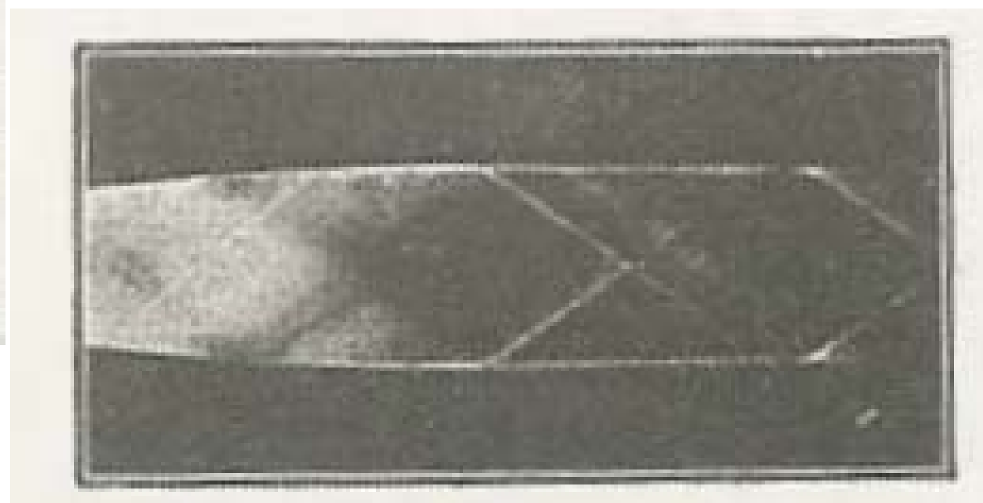
$p_{exit} > p_{back}$
pressure

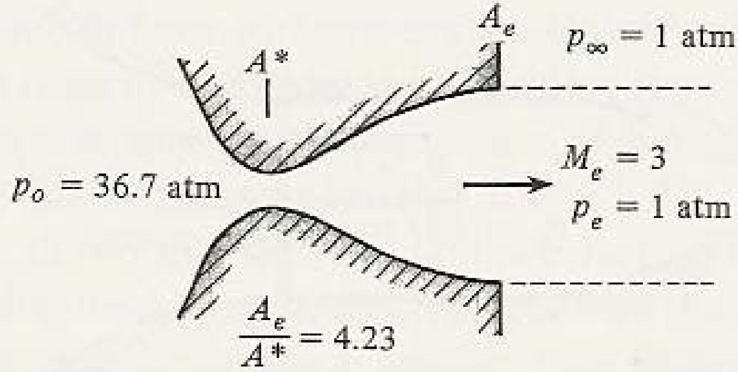
flow compresses into exit region adjustment in a Oblique Shock





NEUTRAL



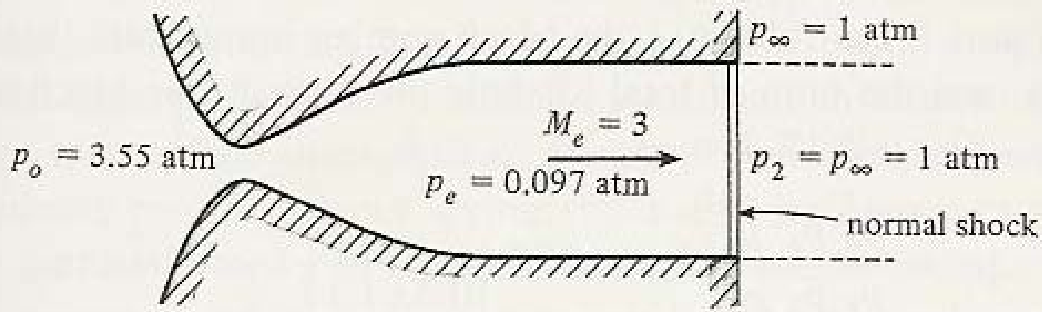


Nozzle exhausting directly to the atmosphere.

Isentropic C.10 @ M = 3;

$$\frac{p_e}{p_O} = .02722$$

$$p_O = 36.73 \times 1\ \text{atm} = 36.73$$



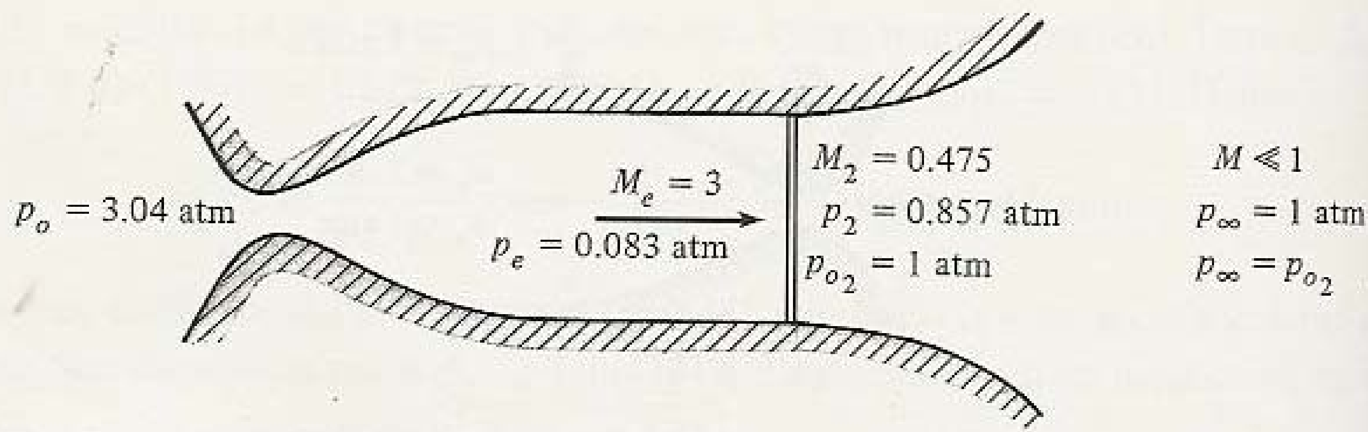
Nozzle with a normal shock at the exit, exhausting to the atmosphere.

Normal Shock C.11 @ $M_1 = 3$;

$$\frac{p_2}{p_1} = 10.33$$

$$p_O = \frac{p_O}{p_1} \times \frac{p_1}{p_2} \times p_2$$

$$p_O = \frac{36.73}{10.33} \times 1 = 3.56\ \text{atm}$$



Nozzle with a normal-shock diffuser. The normal shock is slightly upstream of the divergent duct.

Normal Shock C.11 @ $M_1 = 3$;

$$\frac{p_2}{p_1} = 10.33, M_2 = .4752$$

$$p_1 = p_2 \times \frac{p_1}{p_2} = .855 \times \frac{1}{10.33} = .083 \text{ atm}$$

Isentropic C.10 @ $M_2 = .4752$;

$$\frac{p_2}{p_O} = .854$$

$$p_2 = 1 \text{ atm} \times .854 = .854 \text{ atm}$$

Isentropic @ $M = 3$;

$$\frac{p_1}{p_O} = .02722$$

$$p_O = p_1 \times \frac{p_O}{p_1} = .083 \times 36.73 = 3.049 \text{ atm}$$