STEADY, INVISCID (potential flow, irrotational) INCOMPRESSIBLE

CONTINUITY EQUATION

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Ψ is a solution to the continuity equation where,

$$u=\frac{\partial \Psi}{\partial y} \quad \text{and} \quad v=\frac{\partial \Psi}{\partial x}$$

subsitiuting into the continuity equation,

$$\frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \Psi}{\partial x}\right)=0$$

Ψ is the Stream Function

$\Psi =$ constant is a streamline

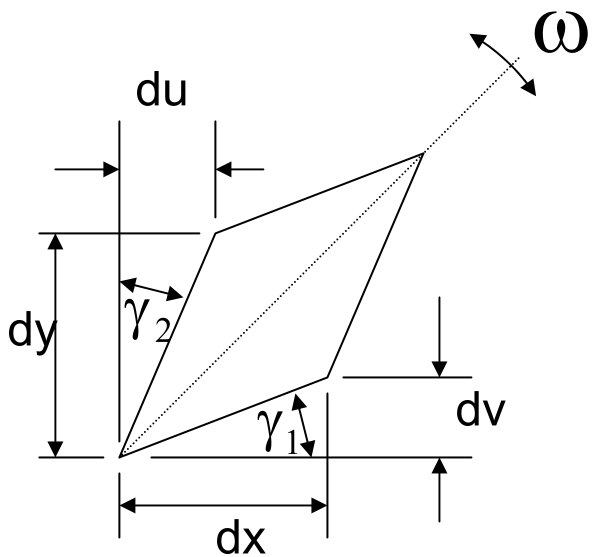
$$\vec{V}=\nabla\Phi$$

$$\vec{V}=\frac{\partial \Phi}{\partial x}\vec{i}+\frac{\partial \Phi}{\partial y}\vec{k}$$

$$\nabla \bullet \vec{V}=0$$

MOMENTUM EQUATION

Bernoulli's equation along a streamline, $\Psi =$ constant



$$\tan\gamma_1 = \frac{du}{dy} = \gamma_1, \qquad \tan\gamma_2 = \frac{dv}{dx} = \gamma_2$$

for small angles $\tan\alpha = \alpha$

$$\omega,\ \text{rotation} = \frac{1}{2}(\gamma_1 + \gamma_2) = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)$$

for irrotational flow, $\omega = 0$,

$$\frac{dv}{dx} = \frac{du}{dy}$$

Φ is a solution to this equation where,

$$u = \frac{\partial\Phi}{\partial x} \quad \text{and} \quad v = \frac{\partial\Phi}{\partial y}$$

subsitiuting to prove this,

$$\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial x}\right)$$

Φ is theVelocity PotentialΦ

ϕ and Ψ are perpendicular

$$d\Phi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy$$

$$d\Phi = u dx + v dy$$

$$\text{where } \Phi \text{ is constant } d\Phi = 0$$

$$\frac{dy}{dx} = -\frac{u}{v}$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$d\psi = -v dx + u dy$$

$$\text{where } \Psi \text{ is constant } d\Psi = 0$$

$$\frac{dy}{dx} = \frac{u}{v}$$

$$\frac{dy}{dx}_{\Phi} \times \frac{dx}{dy}_{\Psi} = -\frac{u}{v} \times \frac{v}{u} = -1$$

LAPLACE's EQUATION

Laplace's equations is of the form,

$$\nabla^2 F = 0$$

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0$$

where F is a function and solution

Laplace's equation is linaer

solutions, F, can be added

$$\frac{\partial^2 F_{\substack{\text{combined}\\\text{solution}}}}{\partial x^2} + \frac{\partial^2 F_{\substack{\text{combined}\\\text{solution}}}}{\partial y^2} = \sum_i \left(\frac{\partial^2 F_i}{\partial x^2} + \frac{\partial^2 F_i}{\partial y^2} \right)$$

$$\Psi_{\substack{\text{combined}\\\text{solution}}} = \sum_i \Psi_i$$

$$\Phi_{\substack{\text{combined}\\\text{solution}}} = \sum_i \Phi_i$$

It can be shown that Ψ and Φ satisfy Laplace's equation

1) substitute the expressions for ψ into the irrotational condition

2) substitute the expressions for Φ into the continuity equation

For Continuity.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

from the definition of Φ,

$$u = \frac{\partial \Phi}{dx} \; v = \frac{\partial \Phi}{dy}$$

substituting,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

For the condition for irrotational flow,

$$\frac{\partial v}{dx} = \frac{\partial u}{dy}$$

from the definition of Ψ,

$$u = \frac{\partial \Psi}{dy}, v = -\frac{\partial \Psi}{dx}$$

subsitiuting,

$$\frac{\partial}{\partial x}\left(-\frac{\partial \Psi}{dx}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \Psi}{dy}\right)$$

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$$

$$\nabla \bullet \vec{V} = 0$$

$$\nabla \bullet \left(\nabla \Phi\right) = \nabla^2 \Phi = 0$$

$$\nabla \bullet \left(\nabla \Psi\right) = \nabla^2 \Psi = 0$$

EXAMPLE

$$\Phi = x^2 + y^2$$

$$\Psi = 2xy$$

$$u = \frac{\partial\Phi}{dx} = 2x \qquad u = \frac{\partial\Psi}{dy} = 2x$$

$$v = \frac{\partial\Phi}{dy} = -2y \qquad v = -\frac{\partial\Psi}{dx} = -2y$$

verify solutions,

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 = 2 - 2 = 0$$

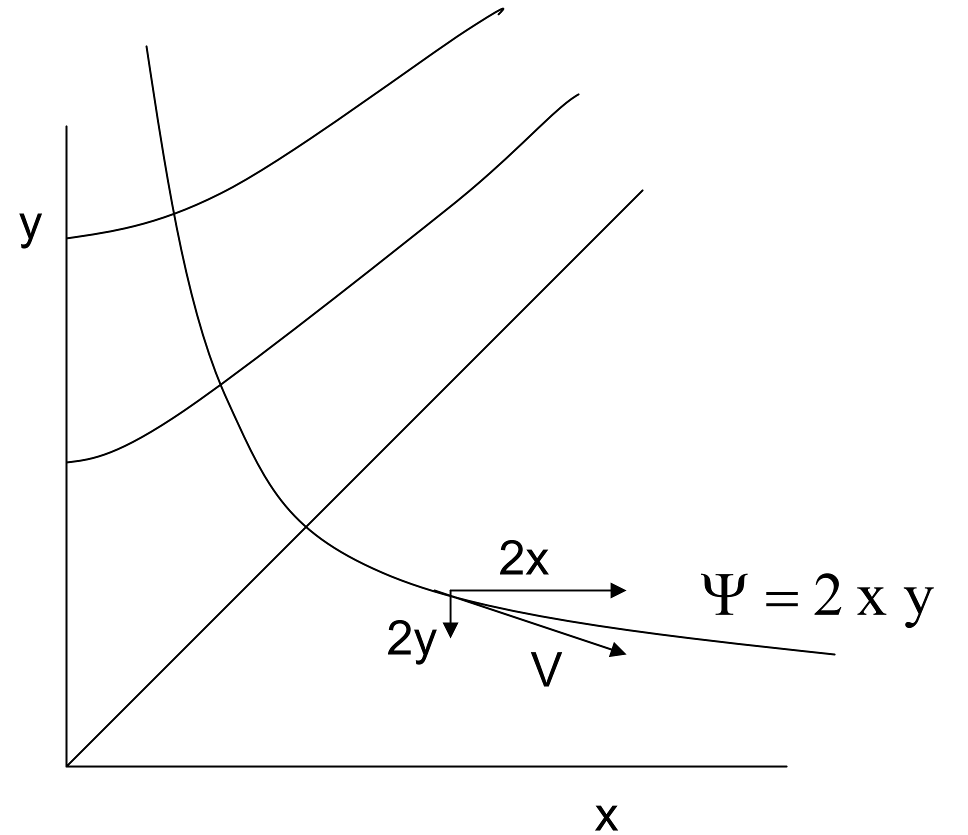
$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0 - 0 = 0$$

Bernoulli's Equation

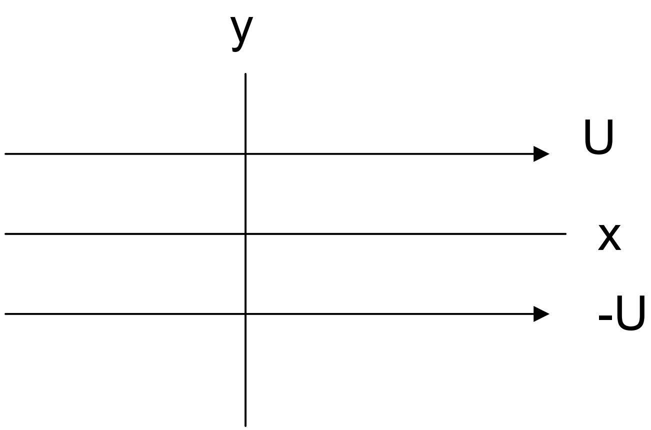
$$p = p_O - \frac{\rho V^2}{2}$$

$$p = p_O - \frac{\rho(u^2 + v^2)}{2}$$

$$p = p_O - 2\rho(x^2 + y^2)$$



UNIFORM FLOW SOLUTION



$u = U, v = 0$

$u = \frac{\partial \Phi}{\partial x} = U,$

integrating,

$\Phi = Ux + \text{constant}$

choose $\Phi = 0, \Psi = 0, \quad$ at $x = 0, y = 0$

$\Phi = Ux$

in cylindrical coordinates where.

$x = r\cos\theta\,, y = r\sin\theta$

$\Phi = U\, r\cos\theta$

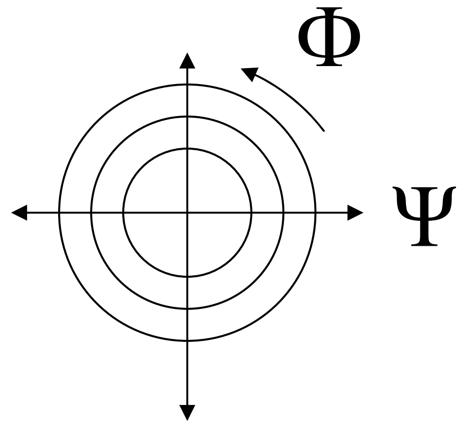
$u = \frac{\partial \Psi}{\partial y} = U$

$\Psi = Uy$

in cylindrical coordinates,

$\Psi = U\, r\sin\theta$

SOURCE FLOW SOLUTION



q = volume flow flux

$$u_R = \frac{q}{2\pi r}, v_\theta = 0$$

$$u_R = \frac{\partial \Phi}{dr}$$

$$\frac{\partial \Phi}{dr} = \frac{q}{2\pi r}$$

$$\Phi = \int \frac{q}{2\pi} dr$$

$$\Phi = \frac{q}{2\pi} \ln r$$

In Cartesian Coordinmates where,

$$r = \sqrt{x^2 + y^2}$$

$$\Phi = \frac{q}{2\pi} \ln \sqrt{x^2 + y^2}$$

$$u_R = \frac{1}{r} \frac{\partial \Psi}{\partial \theta}$$

$$\frac{q}{2\pi r} = \frac{1}{r} \frac{\partial \Psi}{\partial \theta}$$

$$\psi = \frac{q}{2\pi} \theta$$

In Cartesian Coordinates,

$$\text{where } \tan^{-1}\left(\frac{y}{x}\right) = \theta$$

$$\Psi = \frac{q}{2\pi} \tan^{-1}\left(\frac{y}{x}\right)$$

POTENTIAL VORTEX SOLUTION

Potential Vortex

$$ru_\theta = \text{constant}$$

$$u_\theta = \frac{\text{constant}}{r}$$

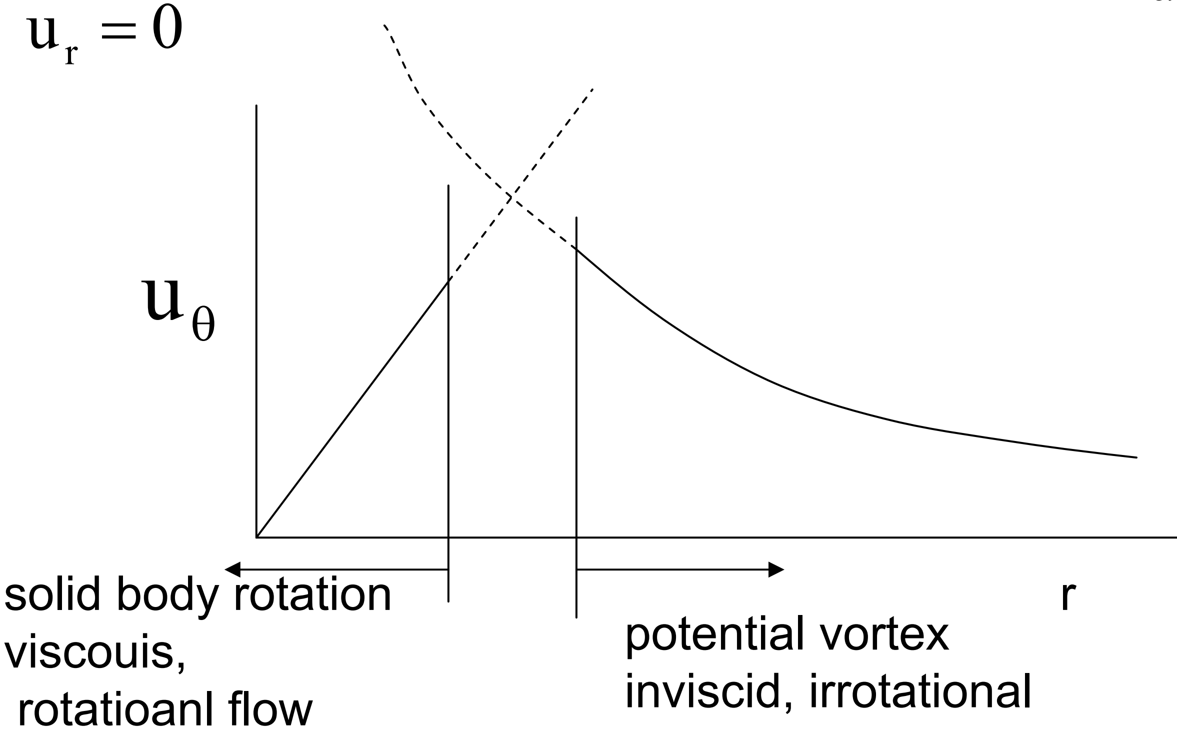
$$u_r = 0$$

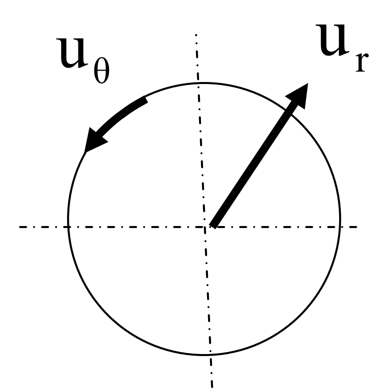
Solid Body Rotation

$$u_\theta = \pi D \omega$$

$$u_\theta = 2\pi r \omega$$

$$u_\theta = C r$$





Potential Vortex

Circulation Γ

$$u_\theta r = \text{constant}$$

$$u_\theta = \frac{\text{constant}}{r}$$

$$\text{circulation}, \Gamma = \oint\!\!\!\oint_S \vec{V} \bullet d\vec{s}$$

$$\Gamma = \int u_\theta r d\theta$$

$$\Gamma = \int_0^{2\pi} \left(\frac{\text{constant}}{r} \right) r d\theta$$

$$\Gamma = 2\pi \times \text{constant}$$

$$\text{constant} = \frac{\Gamma}{2\pi}$$

$$u_\theta = \frac{\Gamma}{2\pi r}$$

Ψ and Φ

$$u_\theta = \frac{d\Phi}{r\, d\theta} = \frac{\Gamma}{2\, \pi\, r}$$

$$\Phi = \int \frac{\Gamma}{2\, \pi\, r}\, r\, d\theta$$

$$\Phi = \frac{\Gamma}{2\, \pi} \theta$$

$$u_\theta = -\frac{d\Psi}{dr} = \frac{\Gamma}{2\, \pi\, r}$$

$$\Psi = \int \frac{\Gamma}{2\, \pi\, r} dr$$

$$\Psi = -\frac{\Gamma}{2\, \pi} \ln r$$

CIRCULATION

intensity of rotation

in a control volume

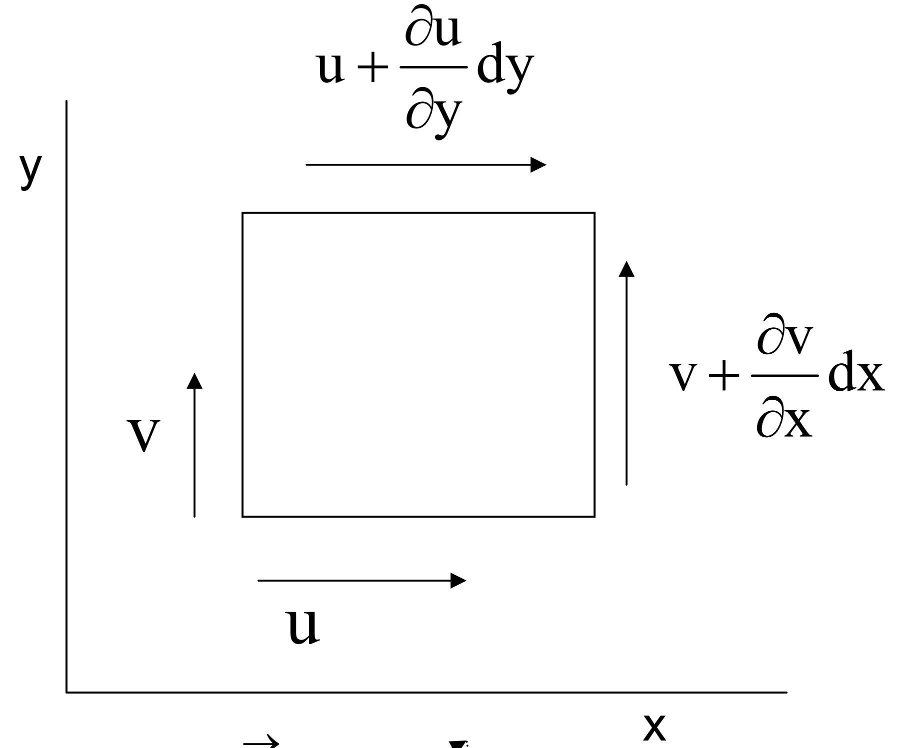
$$\Gamma = \iint_S \vec{V} \bullet d\vec{S}$$

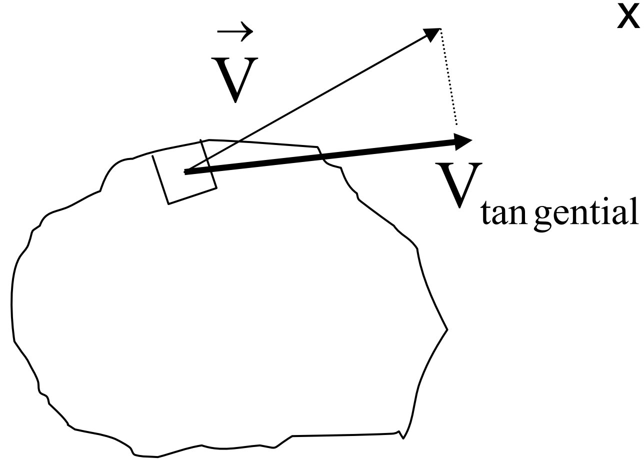
$$\Gamma = \iint_S V_{\substack{\text{tangent to}\\ \text{surface}}} dS$$

$$d\Gamma = u dx + \left(v + \frac{\partial v}{\partial x} dx \right) dy$$

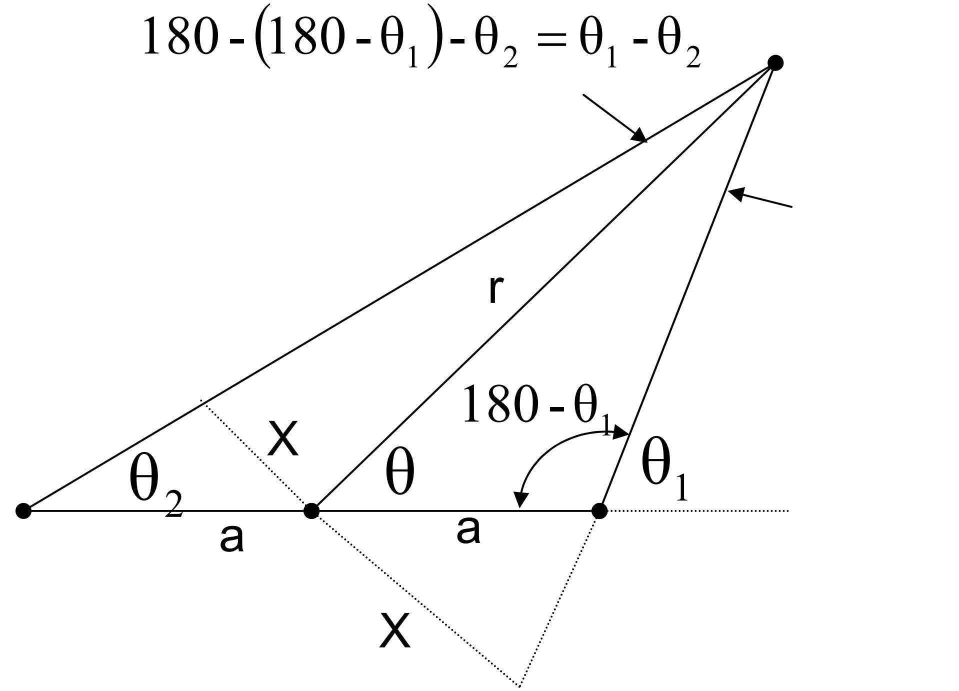
$$- \left(v + \frac{\partial u}{\partial y} dy \right) dx - v dy$$

$$= \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy = 2 \iint_S \omega_z dS$$





DOUBLET Ψ and Φ



$$\sin k,\ \Psi_1 = -\frac{q}{2\pi}\theta_1, \quad \text{source},\ \Psi_2 = \frac{q}{2\pi}\theta_2$$

DOUBLET – sink plus source separated by a distance a

$$\Psi = \Psi_1 + \Psi_2 = -\frac{q}{2\pi}(\theta_1 - \theta_2)$$

$$\tan(\theta_1 - \theta_2) \approx \frac{2X}{r} \quad \text{as} \quad a \to 0$$

$$\text{for small } (\theta_1 - \theta_2), \quad \tan(\theta_1 - \theta_2) = \theta_1 - \theta_2$$

$$\theta_1 - \theta_2 \approx \frac{2X}{r}$$

$$X = a \sin\theta$$

$$\Psi = -\frac{q}{2\pi}\frac{2\,a\sin\theta}{r}$$

$$\Psi = -\frac{K\sin\theta}{r}$$

$$\text{Since } \frac{\partial\Phi}{\partial r} = \frac{1}{r}\frac{\partial\Psi}{\partial\theta}$$

$$\frac{\partial\Phi}{\partial r} = \frac{1}{r}\frac{\partial\left(-\frac{K\sin\theta}{r}\right)}{\partial\theta} = -\frac{K\cos\theta}{r^2}$$

$$\Phi = +\frac{K\cos\theta}{r}$$

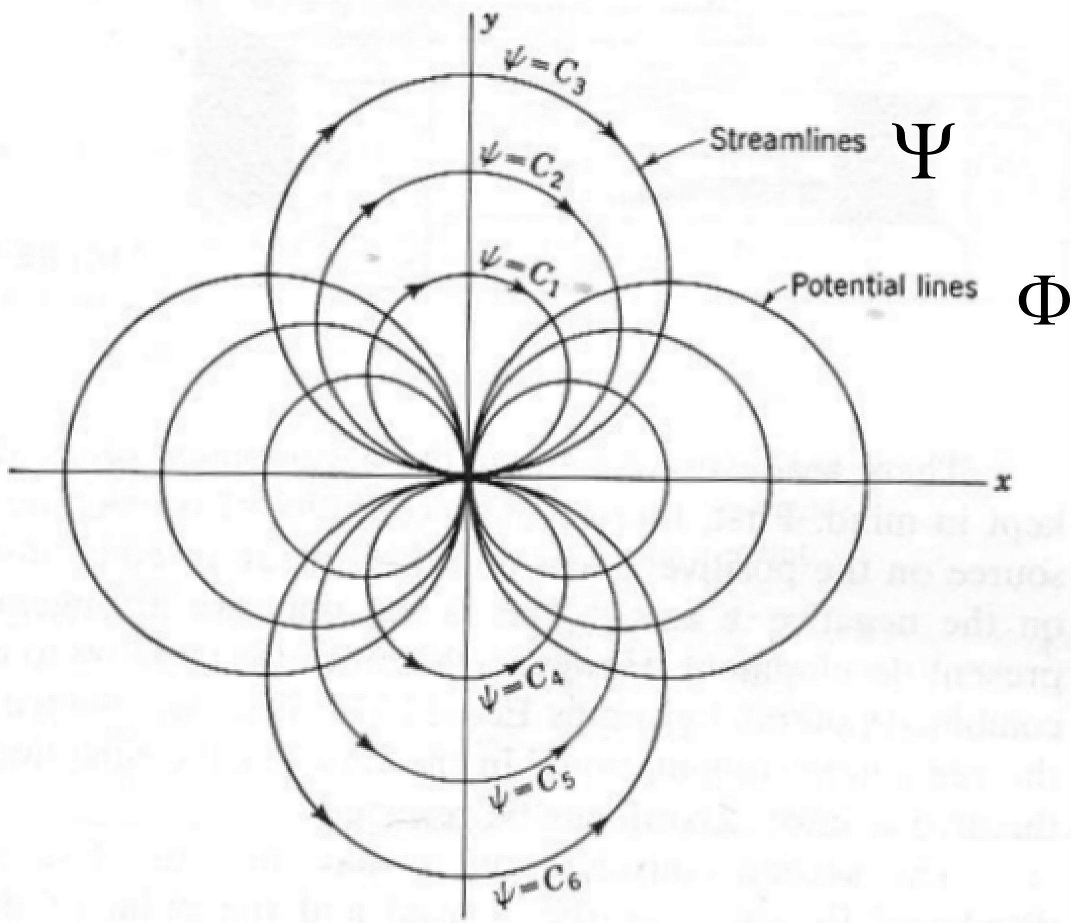
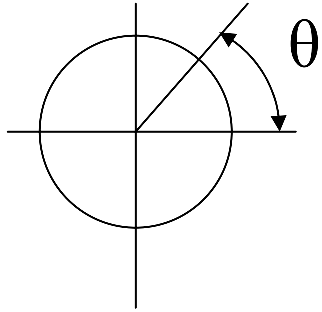


FIGURE 12.28
Flow net for doublet.

DOUBLET PLUS UNIFORM FLOW



$$\psi = \psi_{\substack{\text{uniform}\\ \text{flow}}} + \psi_{\text{doublet}}$$

$$\psi = U_\infty\ r\sin\theta - \frac{K\sin\theta}{r}$$

$\psi = 0$, is a closed surface

$@\ \theta = 0,\ \ \psi = 0$

$@\ \theta = \pi,\ \ \psi = 0$

$$U_\infty r\sin\theta = \frac{K\sin\theta}{r}$$

$$r = \sqrt{\frac{K}{U_\infty}} \Rightarrow \text{a circle of radius } r$$

$K = U_\infty^2 R^2$ doublet of strength K required to get a circle of raddius R

$$\Phi = U_\infty r\cos\theta + \frac{U_\infty R^2\cos\theta}{r}$$

$$u_r = \frac{\partial\Phi}{\partial r} = \frac{\partial}{\partial r}\left(U_\infty r\cos\theta + \frac{U_\infty R^2\cos\theta}{r}\right)$$

$$u_r = U_\infty\cos\theta - \frac{U_\infty R^2\cos\theta}{r^2}$$

$$u_\theta = \frac{\partial\Phi}{r\partial\theta} = \frac{\partial}{r\partial\theta}\left(U_\infty r\cos\theta + \frac{U_\infty R^2\cos\theta}{r}\right)$$

$$u_\theta = -U_\infty\sin\theta - \frac{U_\infty R^2\sin\theta}{r^2}$$

$@\ \theta = 0,\ \ u_\theta = 0$ at the front stagnation point

$@\ \theta = \pi,\ \ u_\theta = 0$ at the rear stagnation point

$@\ r = R$, the surface of the cylinder,

$$u_\theta = -U_\infty\cos\theta - \frac{U_\infty R^2\sin\theta}{r^2}$$

$$u_\theta = -2\,U_\infty\sin\theta$$

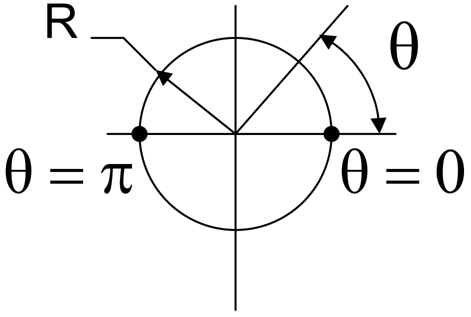
STAGNATION POINTS

stagnation points at $u_\theta = 0, v_r = 0, r = R$

$u_\theta = -2U_\infty \sin\theta$

$u_\theta = 0$ at $\theta = 0$, the trailing point

and $\theta = \pi$, the leading point



$$u_r = U_\infty \cos\theta - \frac{U_\infty R^2 \cos\theta}{r^2}$$

at $r = R$

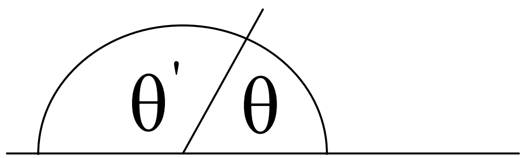
$$u_r = U_\infty \cos\theta - \frac{U_\infty R^2 \cos\theta}{r^2}$$

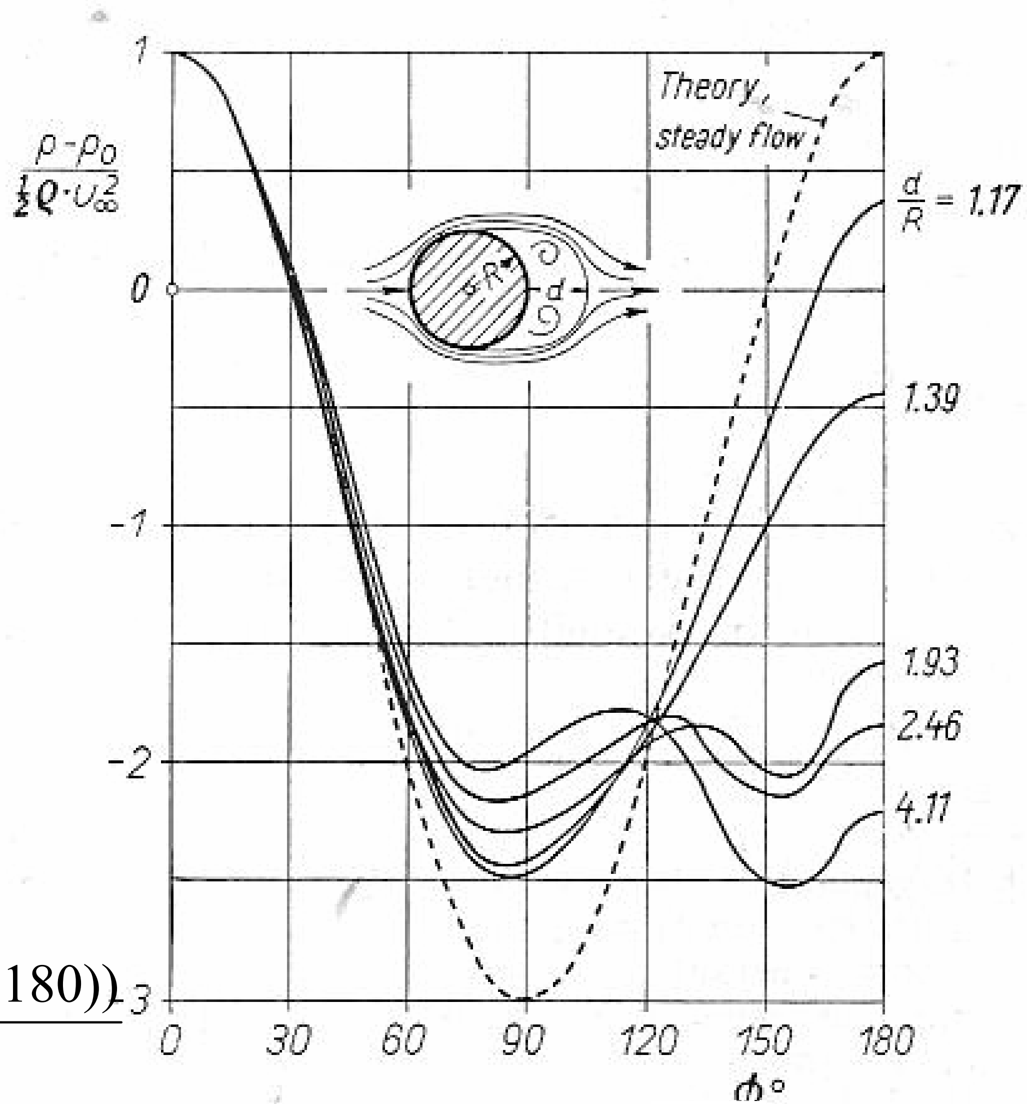
$$u_r = U_\infty \cos\theta - \frac{U_\infty R^2 \cos\theta}{R^2}$$

$$u_r = U_\infty \cos\theta - U_\infty \cos\theta$$

$u_r = 0$ at all θ and $r = R$

Fig. 11.6. Pressure distribution measured around a circular cylinder during the starting process, after M. Schwabe





θ'

d = distance to rear stagnation point

$$u_\theta = -2U_\infty \sin\theta = 2U_\infty \sin(\theta' - 180)$$

Bernoulli's Equation

$$p_\infty + \frac{\rho U_\infty^2}{2} = p_{surface} + \frac{\rho U_{surface}^2}{2}$$

$$p_\infty + \frac{\rho U_\infty^2}{2} = p_{surface} + \frac{\rho (4 U_\infty^2 \sin^2(\theta' - 180))}{2}$$

$$\frac{p_\infty - p_{surface}}{\frac{\rho U_\infty^2}{2}} = \left(1 - 4\sin^2(\theta' - 180)\right)$$

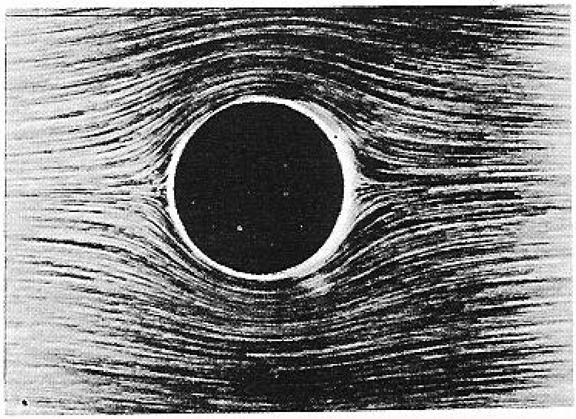


Fig. 11.5a

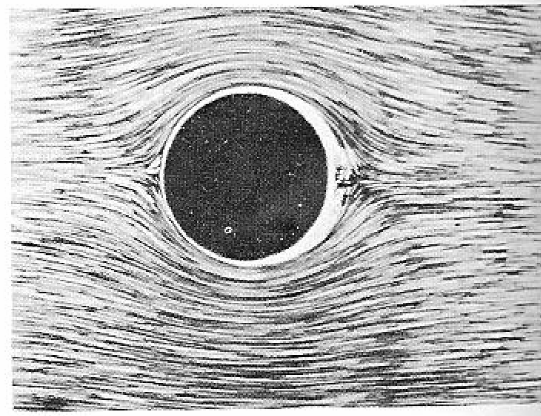


Fig. 11.5b

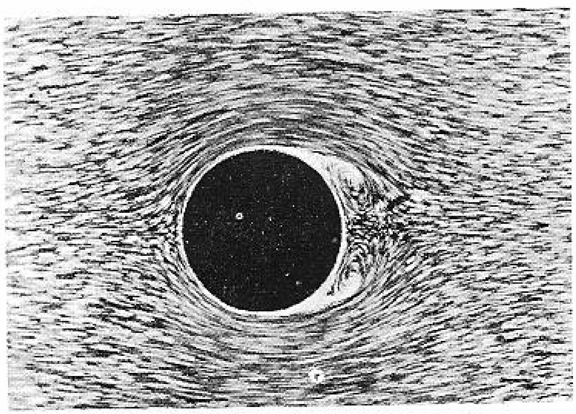


Fig. 11.5c

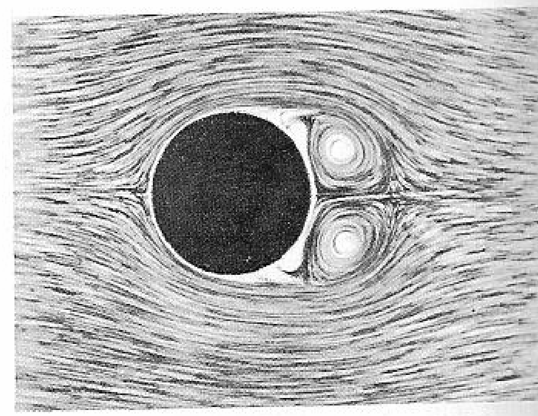


Fig. 11.5d

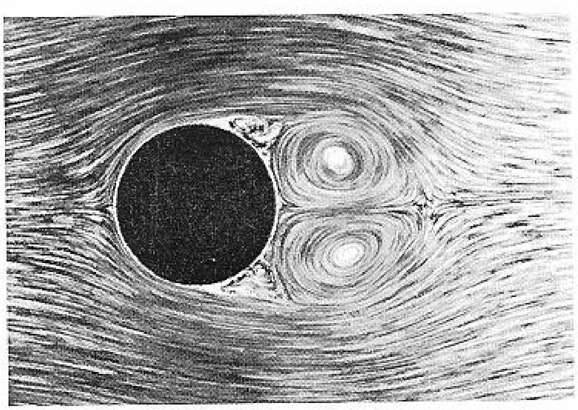


Fig. 11.5e

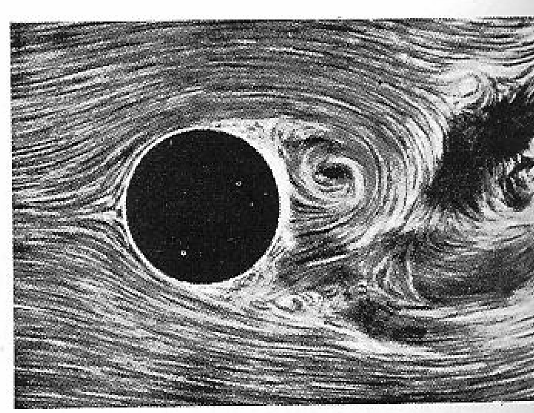


Fig. 11.5f

Fig. 11.5 a to f. Formation of vortices in flow past a circular cylinder after acceleration from rest (L. Prandtl)

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Ideal Flow Machine + Mapper

Instructions | Examples | Source Code | Old Versions

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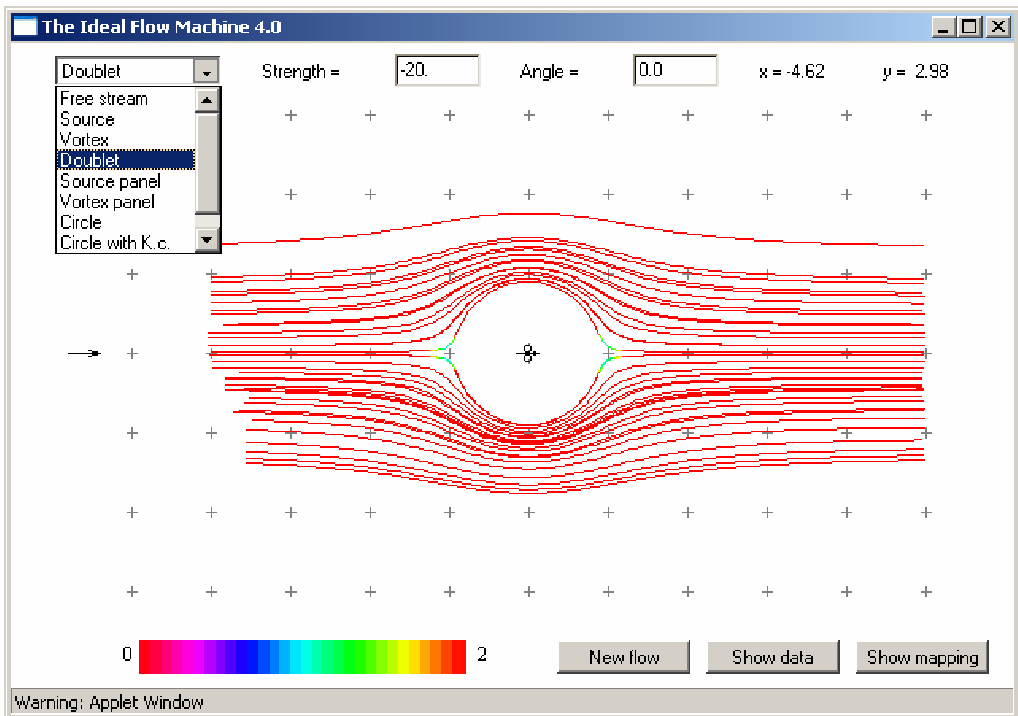
Launch Ideal Flow Machine

An educational Java Applet for those studying elementary ideal flow, Version 4.0

Developed at the Department of Aerospace and Ocean Engineering, Virginia Tech by William J. Devenport

Current Applet Version 4.0. Last HTML/Applet update 8/20/98. Questions or comments please contact William Devenport



DOUBLET IN UNIFORM FLOW

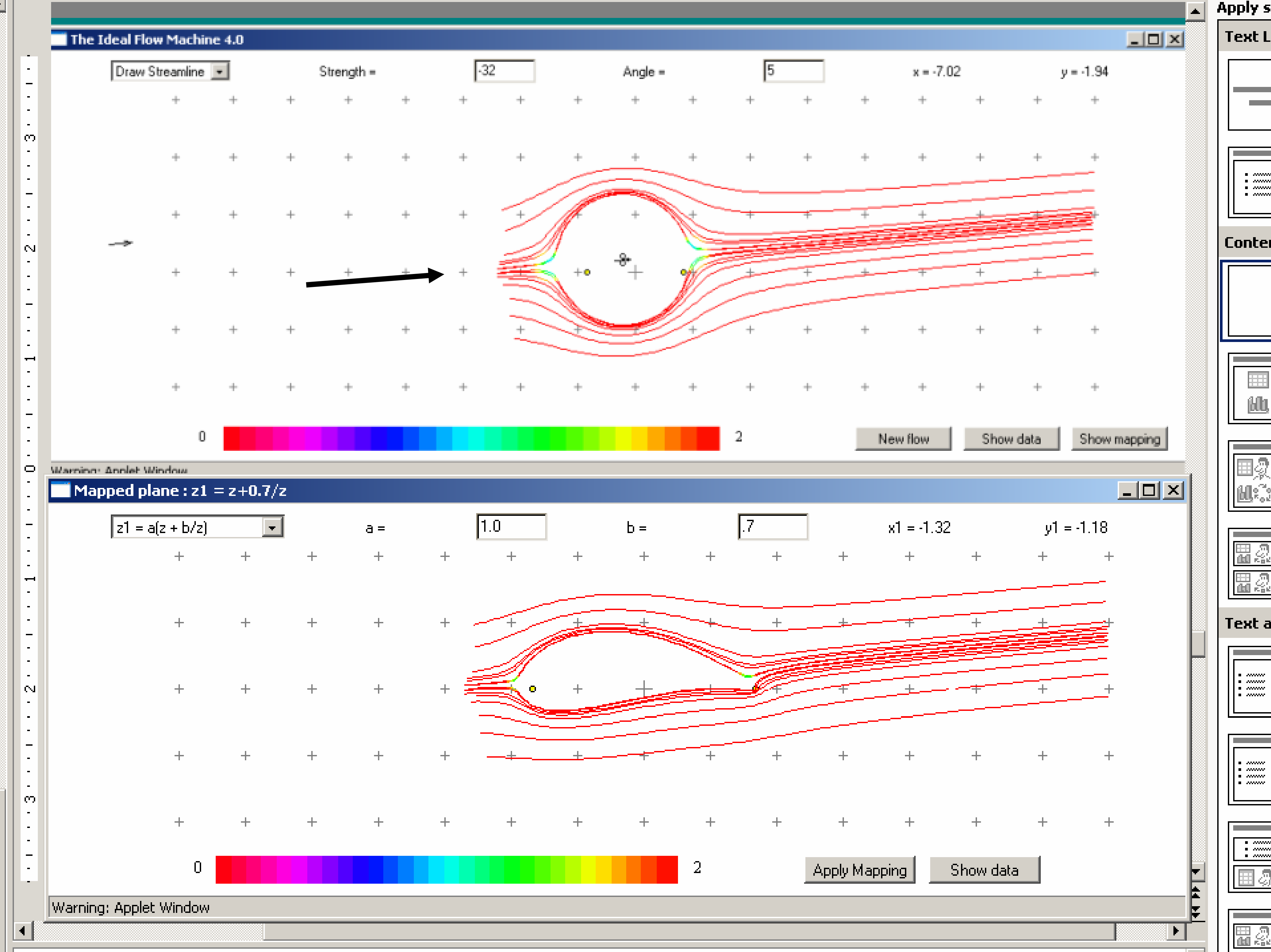
doublet

strength

$$K = r^2 U_\infty^2$$

Jakowski Transformation (of the potential flow Ψ, Φ field)

$$z_1 = a\left(z + \frac{b}{z} \right)$$



The Ideal Flow Machine 4.0
Draw Streamline
Strength =
-32
Angle =
5
x = -7.02
y = -1.94
0
2
New flow
Show data
Show mapping
Warning: Applet Window
Mapped plane : z1 = z+0.7/z
z1 = a(z + b/z)
a =
1.0
b =
.7
x1 = -1.32
y1 = -1.18
0
2
Apply Mapping
Show data
Warning: Applet Window

MAGNUS EFFECT

lift generated by a rotating cylinder in uniform flow

Uniform Flow + Doublet + Potential Vortex

$$\psi = U\, r \sin\theta \; - \frac{K}{2\pi} \ln r \quad - \frac{\Gamma}{2\pi} \ln r$$

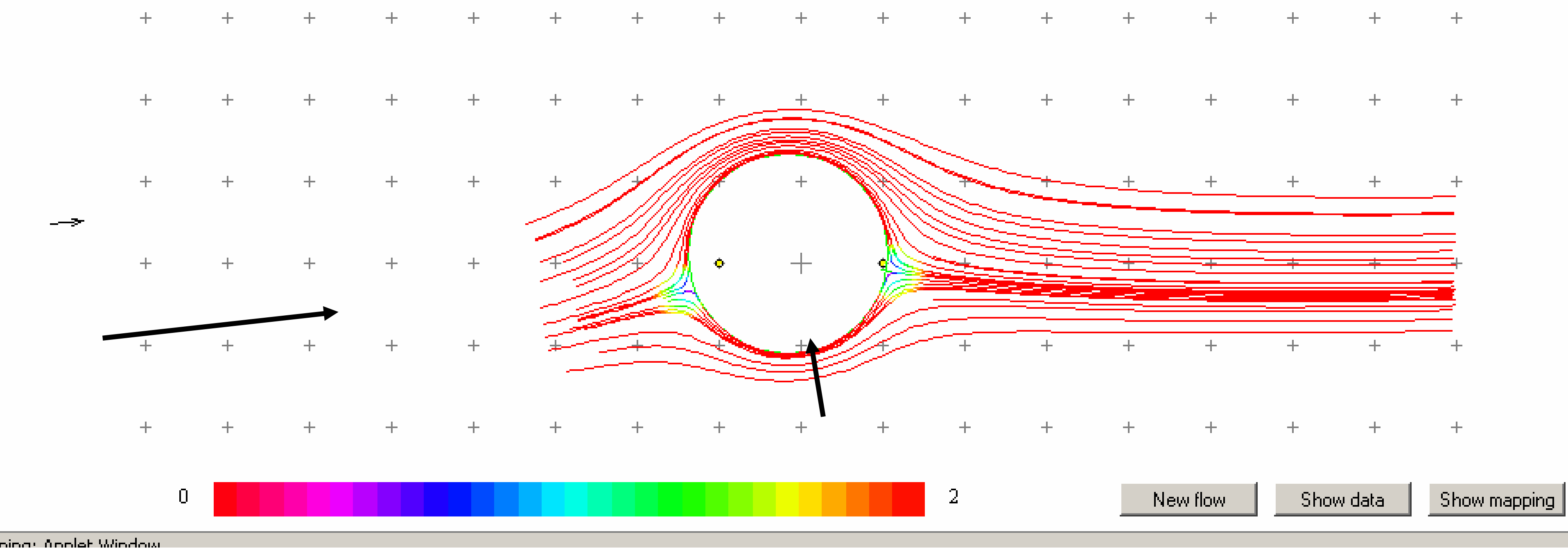
$$\Phi = U\, R \cos\theta \quad + \frac{K \cos\theta}{r} \quad + \frac{\Gamma}{2\pi}\theta$$

K - Doublet strength

Γ – Potential Vortex circulation

$$u_r = \frac{d\Phi}{dx} \qquad\qquad u_\theta = \frac{1}{r}\frac{d\Phi}{d\theta}$$

stagnation points $u_\theta = 0, u_r = 0, r = R$


0
2
New flow
Show data
Show mapping

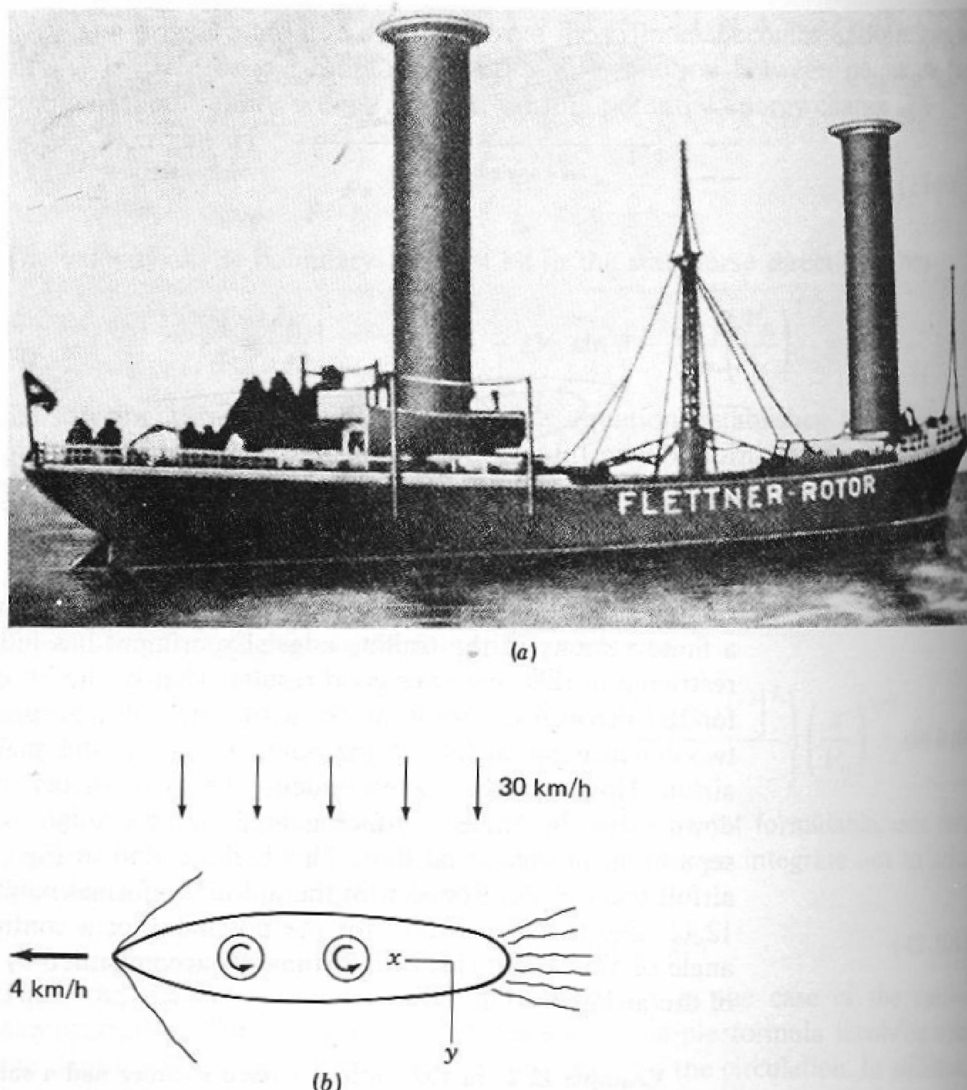
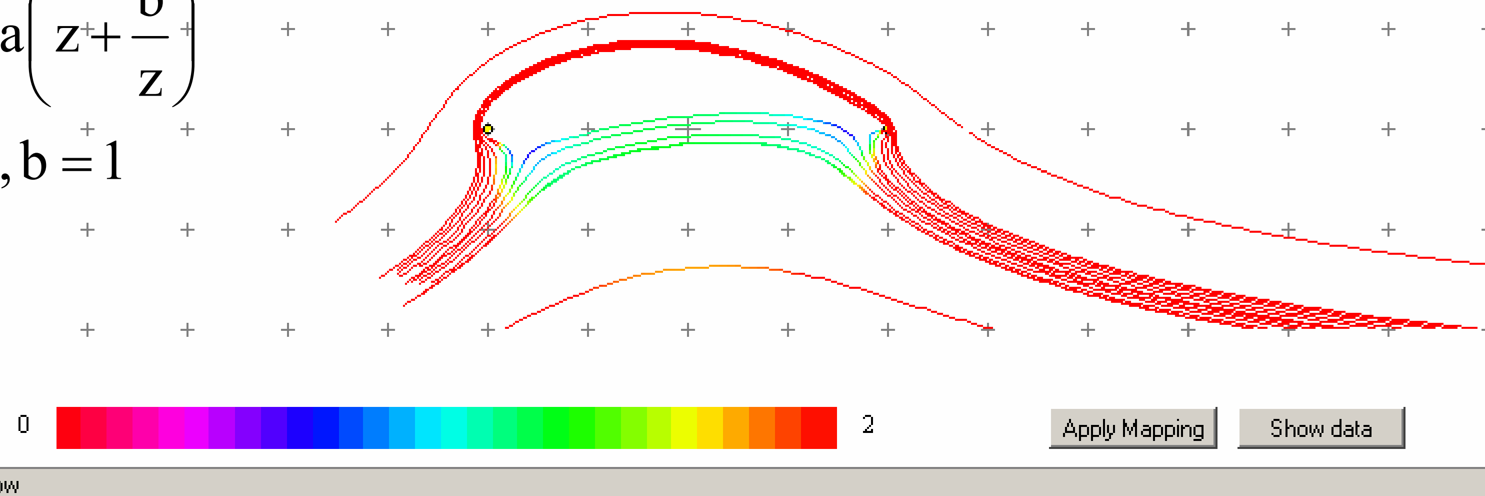


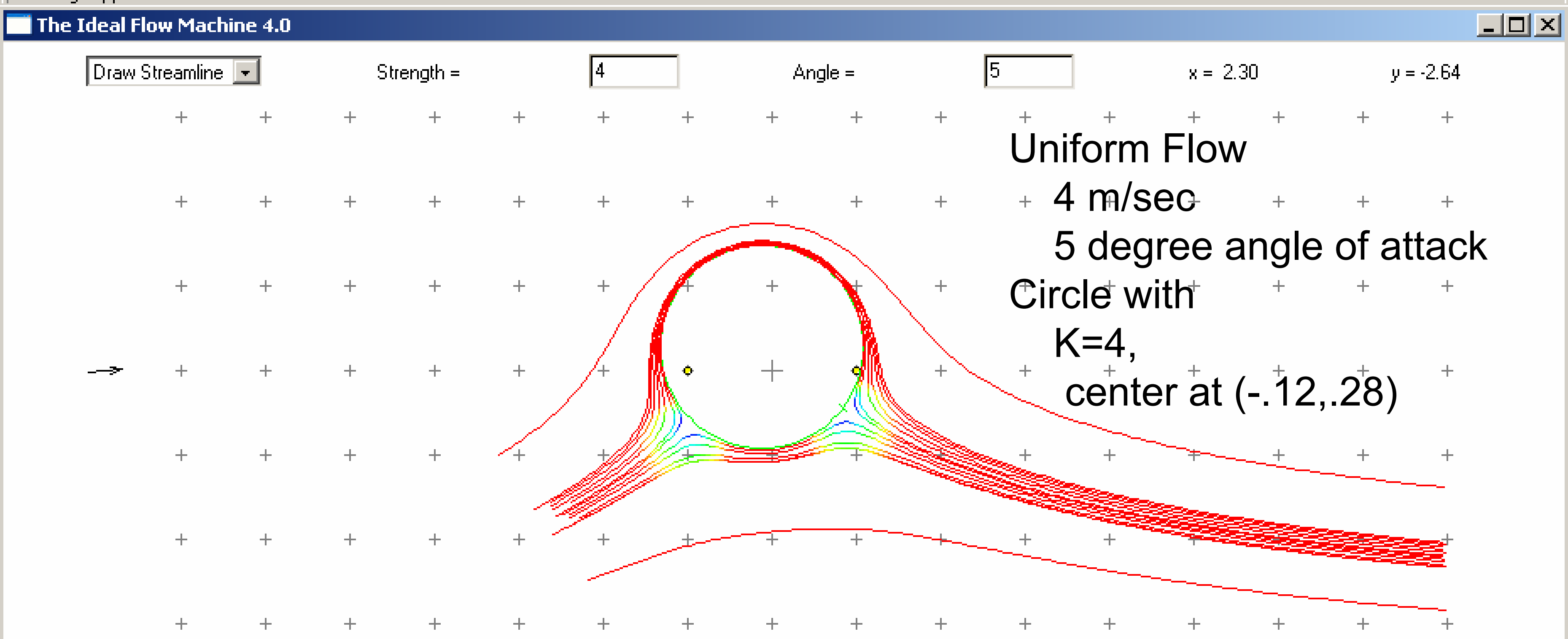
FIGURE 12.43
Flettner's ship. (*From Palmer Coslett, Power from the Wind, New York, Putnam, Van Nostrand Co.*)

Joukowski Transformation

$$z_1 = a\left(z + \frac{b}{z}\right)$$

$$a = 1, b = 1$$





Uniform Flow

4 m/sec

5 degree angle of attack

Circle with

K=4,

center at (-.12,.28)

Kutta Condition

Viscous fluid can not make the sharp trailing edge turn of the indal flow solution.

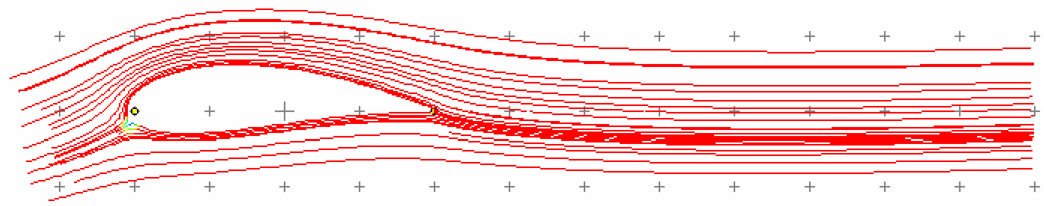
Rear stagnation point must be at the training edge.

Add vortex strength to achieve this condition.

Joukowski Transformation

$$z_1 = a\left(z + \frac{b}{z}\right)$$

$$a = 1, b = 1$$



Uniform Flow

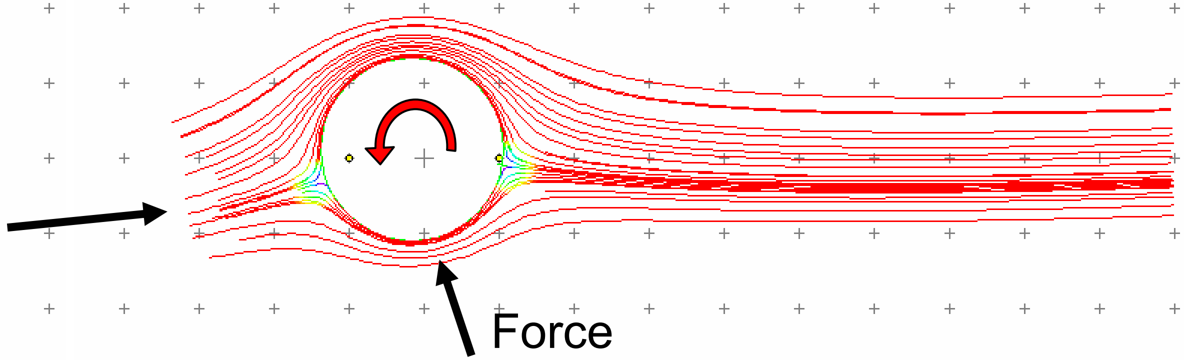
4 m/sec

5 degree angle of attack

Circle with

K=1.4,

center at (-.12,.28)



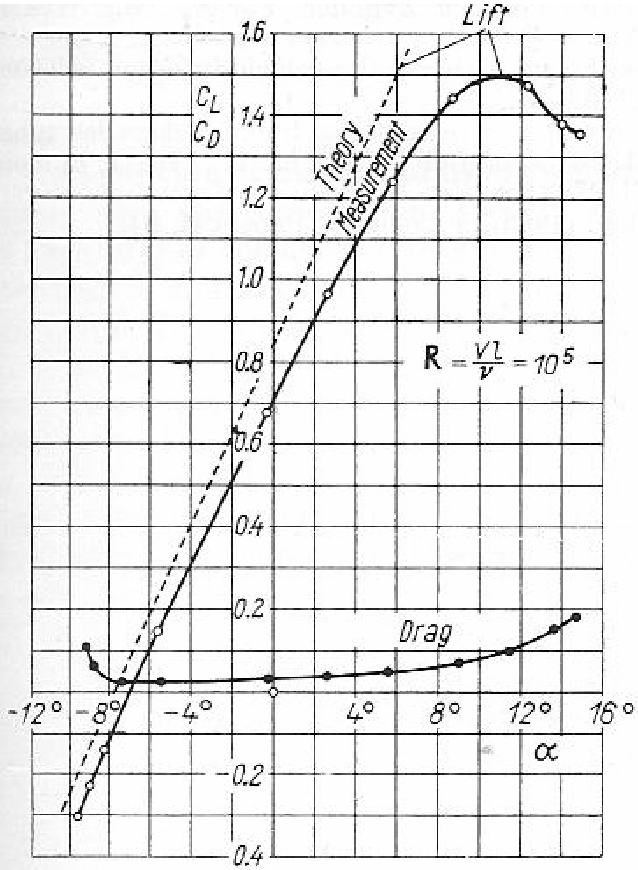


Fig. 1.12. Lift and drag coefficient of a Joukovsky profile in plane flow, as measured by Betz [1]

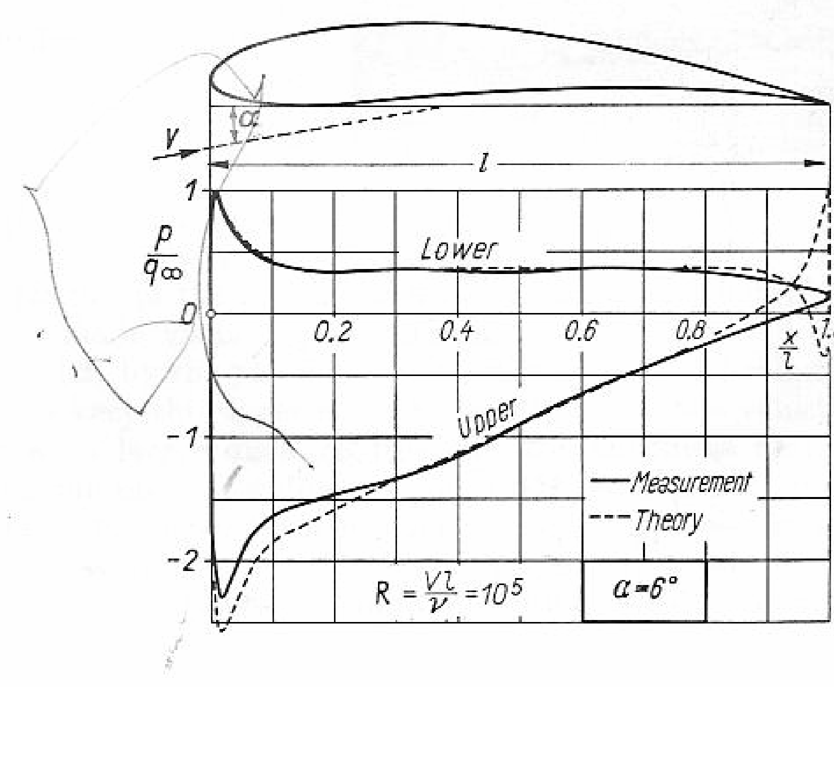


Fig. 1.13. Comparison between the theoretical and measured pressure distribution for a Joukovský profile at equal lifts, after A. Betz [1]