Differential Equations
Chapter 6.1-6.3, 1.4, 1.5

MATH REVIEW

Gauss's Theorem - Divergence Theorem

transforms a surface integral into a volume integral

$$\oiint_A (\vec{n} \bullet \vec{V})\,dA = \oiiint_{vol} (\nabla \vec{V})\,d\,vol \qquad \text{where: } (\vec{V}) \text{ is a vector}$$

$$\oiint_A (a)\,dA = \oiiint_{vol} (\nabla a)\,d\,vol \qquad \text{where: } (a) \text{ is a scalar}$$

Gradient $\nabla = \dfrac{\partial(\ )}{\partial x}\vec{i} + \dfrac{\partial(\ )}{\partial y}\vec{j} + \dfrac{\partial(\ )}{\partial z}\vec{k}$

∇ of a vector is a scalar

∇ of a scalar is a vector

CONTINUITY EQUATION CONSERVATIVE INTEGRAL FORM

Gauss's Theorm transforms a surface integral into a volume integral

$$\oiint_A \vec{n} \bullet \vec{V} dA = \oiiint_{vol} \nabla \vec{V} d\,vol \qquad \text{where, } \nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

Δ (control volume mass) = net mass outflow

$$\frac{\partial}{\partial t} \oiiint_{vol} \rho \; d\,vol = -\oiint_A \left(\rho\, \vec{n} \bullet \vec{V} \right) dA$$

by Smits convention mass inflow is - .

applying Gauss's Theorm to the net mass outflow term,

CONTINUITY EQUATION CONSERVATIVE INTEGRAL FORM

$$\oiiint_{vol} \frac{\partial}{\partial t}\rho \;\; d\,vol \;\; = -\oiiint_{vol} \nabla(\rho\vec{V}) \;\; d\,vol$$

differentiating,

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{V}) = 0 \qquad (6.7)$$

Unsteady, 3 - D, any fluid, variable density

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\,u)}{\partial x} + \frac{\partial(\rho\,v)}{\partial y}\frac{\partial(\rho\,w)}{\partial z} = 0 \qquad (6.6)$$

$$\text{substituting, } \frac{\partial(\rho\,u)}{\partial x} = u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x} \;\; \text{in x, y and z}$$

$$\frac{\partial\rho}{\partial t} + \left(u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}\right) + \left(\rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z}\right) = 0 \quad)$$

Steady, incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

MONENTUM EQUATION CONSERVATIVE INTEGRAL FORM

$$\oiint_A \left(\rho \vec{n} \bullet \vec{V} \, dA \right) \vec{V} + \iiint_{vol} \frac{\partial \left(\rho \vec{V} \right)}{\partial t} d\,vol = \iiint_{vol} \rho\, f \, d\,vol - \oiint_S p \, dS + \oiint_S \tau \, dS$$

using Gauss' s Therom

$$\oiint_A \vec{V} dA = \iiint_{vol} \left(\nabla V \right) d\,vol \quad \text{and} \quad \oiint_S (a) dS = \iiint_{vol} (\nabla a) d\,vol$$

to convert the three surface integrals to volume inteagrals

$$\iiint_{vol} \nabla \left(\rho \vec{V} \right) \vec{V} d\,vol + \iiint_{vol} \frac{\partial \left(\rho \vec{V} \right)}{\partial t} d\,vol = \iiint_{vol} \rho\, f \, d\,vol - \iiint_{vol} \nabla p \, d\,vol + \iiint_{vol} \nabla \tau \, d\,vol$$

differentiating,

$$\frac{\partial \left(\rho \vec{V} \right)}{\partial t} = -\nabla p - \nabla \left(\rho \vec{V} \right) \vec{V} - \nabla \tau + \rho\, f$$

MOMENTUM EQUATIONS

unsteady, 3D, any fluid, variable density

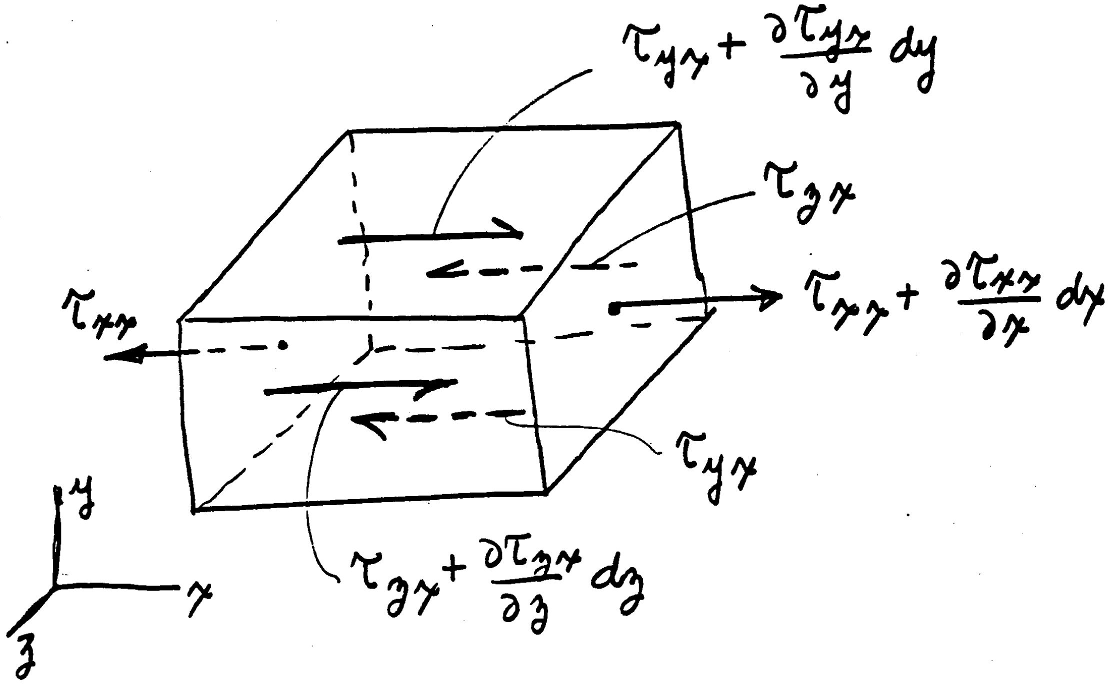
$$\frac{\partial\left(\rho\,\vec{V}\right)}{\partial t} = -\nabla p - \nabla\left(\rho\,\vec{V}\right)\vec{V} - \nabla\tau + \rho\,f$$

$$\frac{\partial}{\partial t}\rho u = -\frac{\partial p}{\partial x} - \left(\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z}\right) - \left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right) + \rho f_x$$

$$\frac{\partial}{\partial t}\rho v = -\frac{\partial p}{\partial y} - \left(\rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z}\right) - \left(\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right) + \rho f_y$$

$$\frac{\partial}{\partial t}\rho w = -\frac{\partial p}{\partial z} - \left(\rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial w}{\partial z}\right) - \left(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{xz} + \frac{\partial}{\partial z}\tau_{zz}\right) + \rho f_z$$

NEWTON'S LAW FOR GENERAL FLOW



$$
\begin{aligned}
dF_x = & \left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} dx \right) dy\, dz - \tau_{xx}\, dy\, dz \\
& + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) dx\, dz - \tau_{yx}\, dx\, dz \\
& + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz \right) dx\, dy - \tau_{zx}\, dx\, dy \\
= & \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) dx\, dy\, dz
\end{aligned}
$$

$$
\rho \frac{Dv_x}{Dt} = \rho B_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}
$$

$$\tau_{ij} = \begin{pmatrix} -p+\tau'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau_{y} & -p+\tau'_{yy} & \tau'_{yz} \\ \tau'_{zx} & \tau'_{zy} & -p+\tau'_{zz} \end{pmatrix}$$

τ'_{ij} DUE TO FRICTION

$$\rho \frac{DV_x}{Dt} = -\frac{\partial p}{\partial x} + \rho B_x + \frac{\partial \tau'_{xx}}{\partial x} + \frac{\partial \tau'_{yx}}{\partial y} + \frac{\partial \tau'_{zx}}{\partial z}$$

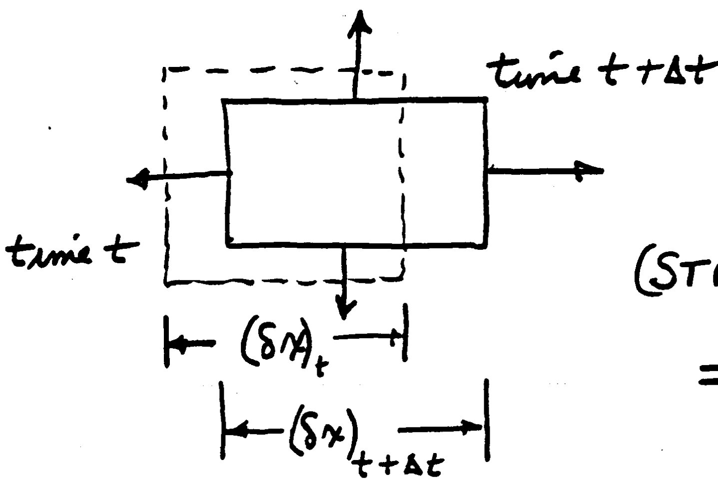
$$\rho \frac{DV_y}{Dt} = -\frac{\partial p}{\partial y} + \rho B_y + \frac{\partial \tau'_{xy}}{\partial x} + \frac{\partial \tau'_{yy}}{\partial y} + \frac{\partial \tau'_{zy}}{\partial z}$$

$$\rho \frac{DV_z}{Dt} = -\frac{\partial p}{\partial y} = \rho B_z + \frac{\partial \tau'_{xz}}{\partial x} + \frac{\partial \tau'_{yz}}{\partial y} + \frac{\partial \tau'_{zz}}{\partial z}$$

NEED TO RELATE τ'_{ij} TO THE
STRAIN FIELD

STRESS + STRAIN

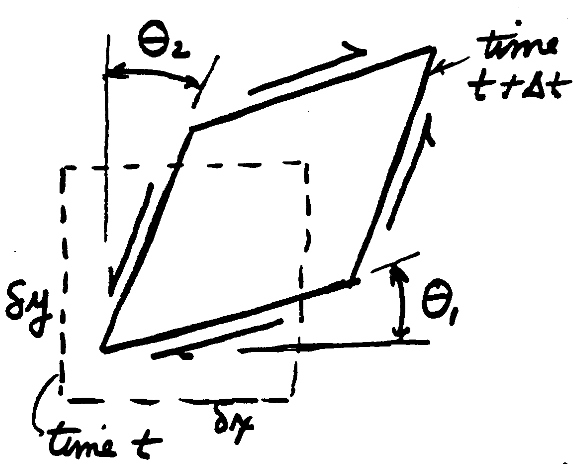
NORMAL STRESS → ELONGATION



$$(\text{STRAIN})_x = \epsilon_{xx}$$

$$= \frac{(\delta x)_{t+\Delta t} - (\delta x)_t}{(\delta x)_t}$$

SHEAR STRESS → CHANGE IN ANGLE



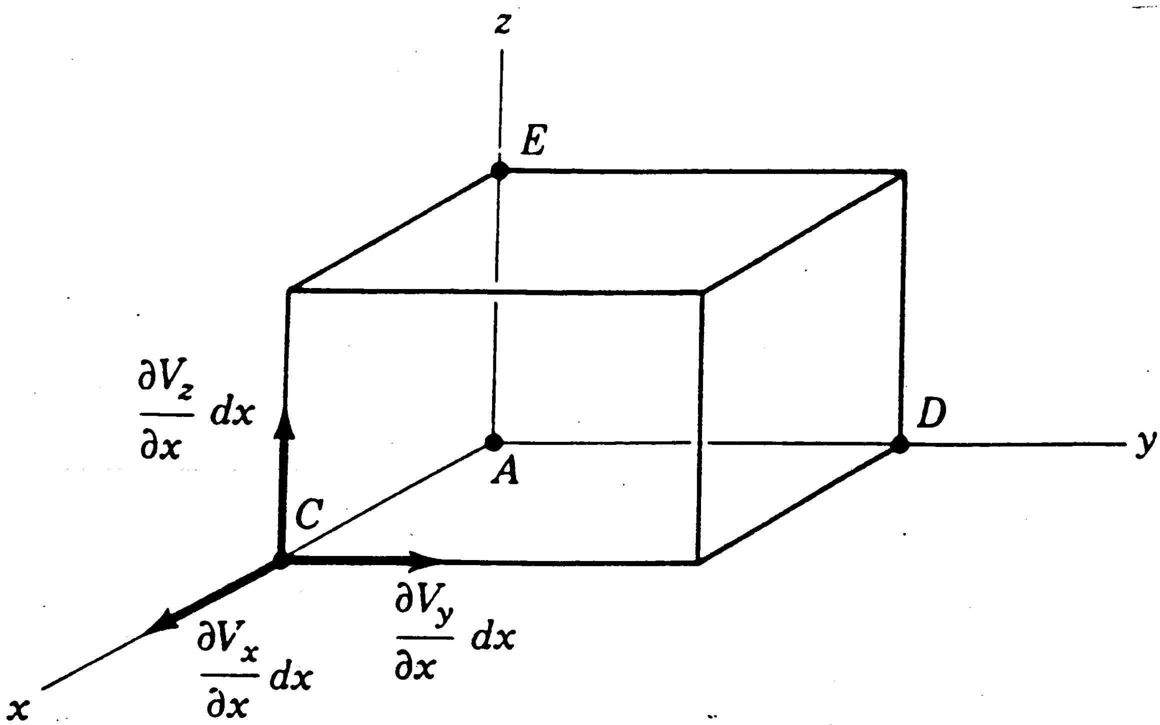
$$(\text{STRAIN})_{xy} = \epsilon_{xy}$$

$$= \theta_1 + \theta_2$$

$$\tau_{xy} \propto \epsilon_{xy}$$

$$(\text{Rotation})_z = \tfrac{1}{2}(\theta_1 - \theta_2)$$

STRAIN + ROTATION



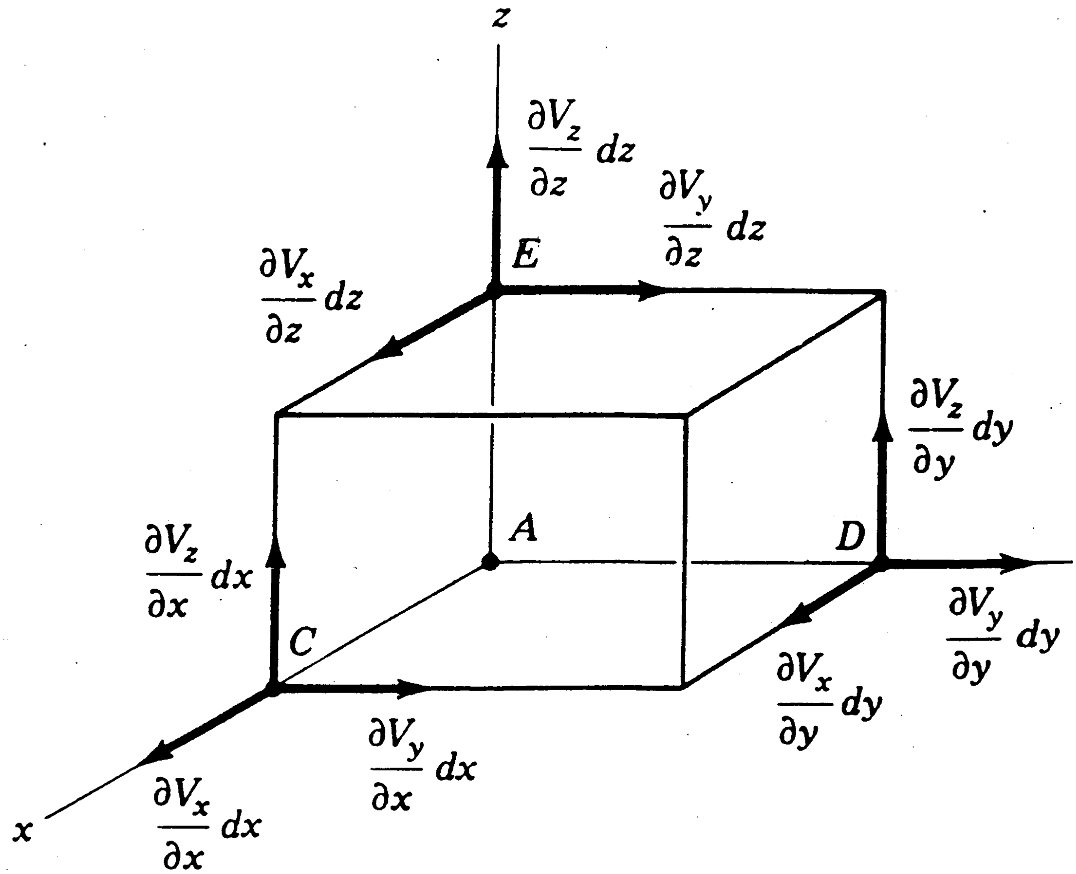
$$\vec{V}_C = \vec{V}_A + \frac{\partial \vec{V}}{\partial x} dx$$

$$(\vec{V}_C - \vec{V}_A) = \frac{\partial V_x}{\partial x} dx\,\hat{\imath} + \frac{\partial V_y}{\partial x} dx\,\hat{\jmath} + \frac{\partial V_z}{\partial x} dx\,\vec{k}$$

$\frac{\partial V_x}{\partial x}$ ELONGATES $\overline{AC}$

$\frac{\partial V_z}{\partial x}$ ROTATES $\overline{AC}$ ABOUT THE Y AXIS

$\frac{\partial V_y}{\partial x}$ ROTATES $\overline{AC}$ ABOUT THE Z AXIS



$$\dot\epsilon_{xx} = \frac{\partial V_x}{\partial x} \qquad \dot\epsilon_{yy} = \frac{\partial V_y}{\partial y} \qquad \dot\epsilon_{zz} = \frac{\partial V_z}{\partial z}$$

$\angle DAE$ $\quad \dot\gamma_{yz} = \frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y} \qquad \omega_x = \frac{1}{2}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)$

$\angle CAE$ $\quad \dot\gamma_{xz} = \frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x} \qquad \omega_y = \frac{1}{2}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)$

$\angle CAD$ $\quad \dot\gamma_{xy} = \frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \qquad \omega_z = \frac{1}{2}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$

EQUATIONS OF MOTION

$$\rho \frac{DV_x}{Dt} = -\frac{\partial p}{\partial x} + \rho B_x + \frac{\partial \tau'_{xx}}{\partial x} + \frac{\partial \tau'_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\rho \frac{DV_y}{Dt} = -\frac{\partial p}{\partial y} + \rho B_y + \frac{\partial \tau'_{xy}}{\partial x} + \frac{\partial \tau'_{yy}}{\partial y} + \frac{\partial \tau'_{zy}}{\partial z}$$

$$\rho \frac{DV_z}{Dt} = -\frac{\partial p}{\partial z} + \rho B_z + \frac{\partial \tau'_{xz}}{\partial x} + \frac{\partial \tau'_{yz}}{\partial y} + \frac{\partial \tau'_{zz}}{\partial z}$$

INCOMPRESSIBLE CONTINUITY

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0$$

STRESS - STRAIN

$$\tau'_{xx} = 2\mu \frac{\partial V_x}{\partial x} \qquad \tau'_{xy} = \tau'_{yx} = \mu\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right)$$

$$\tau'_{yy} = 2\mu \frac{\partial V_y}{\partial y} \qquad \tau'_{xz} = \tau'_{zx} = \mu\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right)$$

$$\tau'_{zz} = 2\mu \frac{\partial V_z}{\partial z} \qquad \tau'_{yz} = \tau'_{zy} = \mu\left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right)$$

NAVIER-STOKES EQUATIONS

FOR AN INCOMPRESSIBLE FLUID

$$\rho \frac{DV_x}{Dt} = -\frac{\partial p}{\partial x} + \rho B_x + \mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right)$$

$$\rho \frac{DV_y}{Dt} = -\frac{\partial p}{\partial y} + \rho B_y + \mu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right)$$

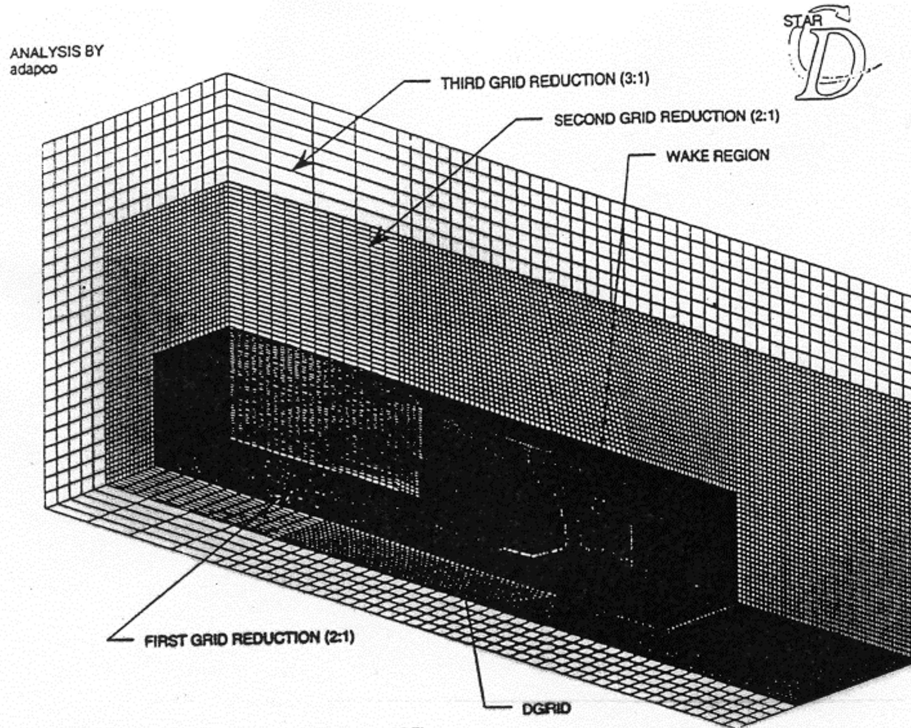
$$\rho \frac{DV_z}{Dt} = -\frac{\partial p}{\partial z} + \rho B_z + \mu\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right)$$

CONTINUITY: $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0$

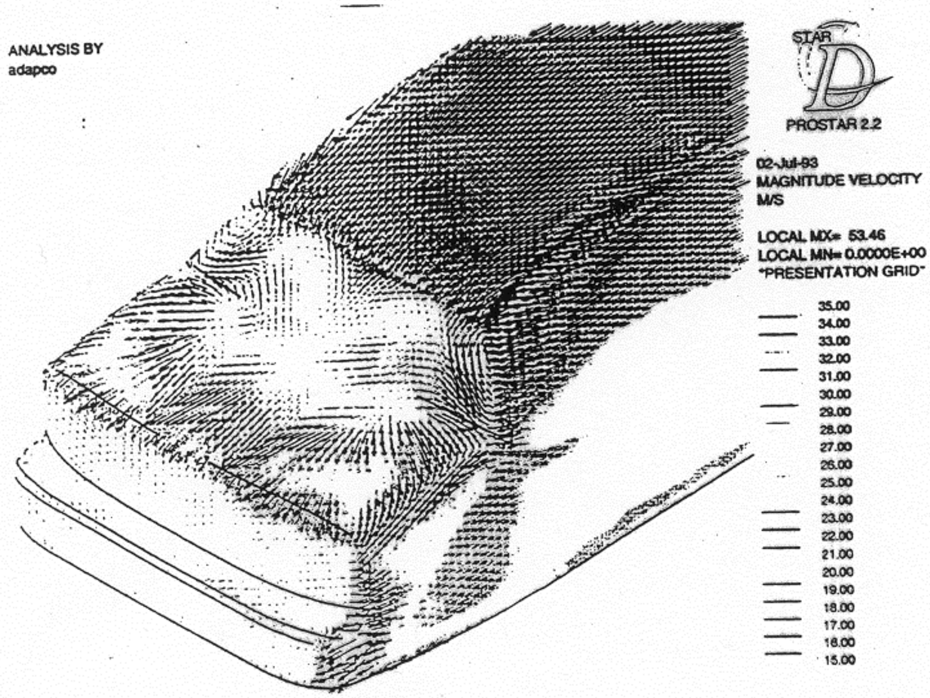
TO BE SOLVED FOR

$V_x, V_y, V_z + p$

AS FUNCTIONS OF x, y, z, t



NOTCHBACK WIND TUNNEL AERODYNAMIC STUDY MODEL
COMPLETE MODEL DOMAIN



WIND TUNNEL AERODYNAMICS STUDY OF NOTCHBACK TEST SHAPE
KE RESULTS - KE TURBULENCE MODEL WITH LUD
VELOCITY MAGNITUDE NEAR THE VEHICLE

VIEW FROM REAR

NUMERICAL SOLUTION

restricting the momentum equation to Newtonian fluids for which the fluids stress is a linear function of the rate of deformation of the fluid - the change of velocity with distance.

for 1 D, $\tau = \mu \frac{du}{dx}$

$$\tau_{xx} = -2\mu \frac{\partial u}{\partial x} + \frac{2}{3}\mu\left(\nabla \vec{V}\right)$$

$$\tau_{yy} = -2\mu \frac{\partial v}{\partial x} + \frac{2}{3}\mu\left(\nabla \vec{V}\right)$$

$$\tau_{zz} = -2\mu \frac{\partial w}{\partial x} + \frac{2}{3}\mu\left(\nabla \vec{V}\right)$$

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

ENERGY EQUATION CONSERVATIVE INTEGRAL FORTM

$$\text{First Law } Q = \Delta E + W = \Delta E + W_{shaft} + W_{viscous} + W_{pressure} + W_{body}$$

$$\text{Work} = \text{Force} \times \text{Velocity}$$

$$W_{shaft} = 0$$

$$\text{Work}_{pressure} \quad -\oiint_A (p\, dA)\vec{V}$$

$$\text{Work}_{body} \quad \iiint_{Vol} (\rho\, f\, d\, vol)\vec{V}$$

$$\text{Work}_{viscous} \, -\oiint_A (\tau\, dA)\vec{V}$$

$$\text{Net Energy into control volume} \qquad \oiint_A (\rho\, V\, dA)\left(e + \frac{V^2}{2}\right)$$

$$\text{Change in energy inside the control volume} \quad \frac{\partial}{\partial t}\iiint_{Vol} \rho\left(e + \frac{V^2}{2}\right) d\, vol$$

$$\text{Heat addition} \quad \oiint_A \vec{q}\, dA$$

$$\text{Internal energy}, \quad U = c_v T$$

First Law $Q = \Delta E + W = \Delta E + W_{\text{shaft}} + W_{\text{viscous}} + W_{\text{pressure}} + W_{\text{body}}$

$$Q = \Delta E_{\substack{\text{net in}\\ \text{control}\\ \text{volume}}} + \Delta E_{\substack{\text{change in}\\ \text{control volume}}} \quad W_{\text{shaft}} + W_{\text{viscous}} + W_{\text{pressure}} + W_{\text{body}}$$

$$Q = \oiint\limits_{A} \left(\rho \vec{\nabla}\, dA\right)\left(e + \frac{V^2}{2}\right) + \frac{\partial}{\partial t} \iiint\limits_{vol} \rho \left(e + \frac{V^2}{2}\right) d\,vol + -\oiint\limits_{A} (\tau\, dA)\vec{V} - \oiint\limits_{A} \left(p\, \vec{V}\right) dA + \iiint\limits_{vol} (\rho\, f\, d\,vol)\vec{V}$$

$$\frac{\partial}{\partial t}\rho\left(c_v T + \frac{V^2}{2}\right) = -\left(\nabla \rho\, \vec{\nabla}\left(c_v T + \frac{V^2}{2}\right)\right) - \nabla \bullet q - \nabla \bullet p\vec{V} - \nabla \bullet (\tau \bullet \vec{V}) + \rho(g \bullet \vec{V})$$

$$\frac{\partial}{\partial t}\rho\left(c_v T + \frac{V^2}{2}\right) = -\left(\frac{\partial}{\partial x} u\rho\left(c_v T + \frac{V^2}{2}\right) + \frac{\partial}{\partial y} w\rho\left(c_v T + \frac{V^2}{2}\right) + \frac{\partial}{\partial y} w\rho\left(c_v T + \frac{V^2}{2}\right)\right)$$

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) - \left(\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w\right)$$

$$-\left(\frac{\partial}{\partial x}\left(\tau_{xx} u + \tau_{xy} v + \tau_{xz} w\right) + \frac{\partial}{\partial y}\left(\tau_{yx} u + \tau_{yy} v + \tau_{yz} w\right) + \frac{\partial}{\partial z}\left(\tau_{zx} u + \tau_{zy} v + \tau_{zz} w\right)\right)$$

$$\rho c_v\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) - T\left(\frac{\partial p}{\partial T}\right)_p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$-\left(\tau_{xx}\frac{\partial u}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}\right) - \left(\tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \tau_{xz}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \tau_{yz}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right)$$

EQUATION SUMMARY - 3D, viscous, variable density

CONTINUITY

$$\frac{\partial\rho}{\partial t}+\left(u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\right)+\left(\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}+\rho\frac{\partial w}{\partial z}\right)=0$$

MOMENTUM $-$ x, y, z directions

$$\frac{\partial}{\partial t}\rho u=-\frac{\partial p}{\partial x}-\left(\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xx}+\frac{\partial}{\partial y}\tau_{yx}+\frac{\partial}{\partial z}\tau_{zx}\right)+\rho f_x$$

$$\frac{\partial}{\partial t}\rho v=-\frac{\partial p}{\partial y}-\left(\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial y}\tau_{yy}+\frac{\partial}{\partial z}\tau_{zy}\right)+\rho f_y$$

$$\frac{\partial}{\partial t}\rho w=-\frac{\partial p}{\partial z}-\left(\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial w}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xz}+\frac{\partial}{\partial y}\tau_{xz}+\frac{\partial}{\partial z}\tau_{zz}\right)+\rho f_z$$

ENERGY

$$\rho c_v\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)-T\left(\frac{\partial p}{\partial T}\right)_p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$-\left(\tau_{xx}\frac{\partial u}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}\right)-\left(\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\tau_{xz}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\tau_{yz}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)$$

<u>EQUATION SUMMARY</u> - 3D, viscous, variable density

CONTINUITY

$$\frac{\partial \rho}{\partial t}+\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right)+\left(\boxed{\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}}+\rho\frac{\partial w}{\partial z}\right)=0$$

2D steady incompressible,
inviscid -> T= constant, adiabatic

MOMENTUM $-$ x, y, z directions

$$\frac{\partial}{\partial t}\rho u=-\frac{\partial p}{\partial x}-\left(\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xx}+\boxed{\frac{\partial}{\partial y}\tau_{yx}}+\frac{\partial}{\partial z}\tau_{zx}\right)+\rho f_x$$

$$\frac{\partial}{\partial t}\rho v=-\frac{\partial p}{\partial y}-\left(\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial y}\tau_{yy}+\frac{\partial}{\partial z}\tau_{zy}\right)+\rho f_y$$

$$\frac{\partial}{\partial t}\rho w=-\frac{\partial p}{\partial z}-\left(\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial w}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xz}+\frac{\partial}{\partial y}\tau_{xz}+\frac{\partial}{\partial z}\tau_{zz}\right)+\rho f_z$$

ENERGY

$$\rho c_v\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)-T\left(\frac{\partial p}{\partial T}\right)_p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$-\left(\tau_{xx}\frac{\partial u}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}\right)-\left(\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\tau_{xz}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\tau_{yz}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)$$

BOUNDARY LAYER   Prandtl 1904

Divide a flow into two regions according to the forces that prevail

BOUNDARY LAYER

thin layer near wall

viscous forces as improtant as interial forces

$\frac{\partial u}{\partial y}$ large, $\tau = \mu \frac{\partial u}{\partial y}$ very large

ignore traverse momentum equations

2 – D incompresible boundary layer equations,

$$u \frac{\partial u}{\partial y} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial \tau_{yx}}{dy}$$

$$u \frac{\partial u}{\partial y} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\mu}{\rho}\frac{\partial^2 u}{dy^2}$$

$$\frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = 0$$

FREE STREAM

$\tau = 0$, $\mu = 0$,

Potential Flow

isentropic, frictionless

irrotational,

uniform and parallel

EQUATION SUMMARY - 3D, viscous, variable density

CONTINUITY

$$\frac{\partial \rho}{\partial t}+\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right)+\left(\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}+\rho\frac{\partial w}{\partial z}\right)=0$$

2D, Inviscid, steady, compressible

MOMENTUM – x, y, z directions

$$\frac{\partial}{\partial t}\rho u=-\frac{\partial p}{\partial x}-\left(\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xx}+\frac{\partial}{\partial y}\tau_{yx}+\frac{\partial}{\partial z}\tau_{zx}\right)+\rho f_x$$

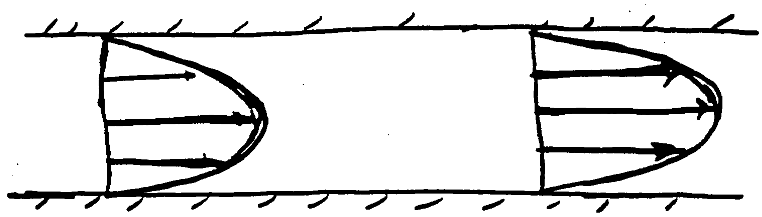
$$\frac{\partial}{\partial t}\rho v=-\frac{\partial p}{\partial y}-\left(\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial y}\tau_{yy}+\frac{\partial}{\partial z}\tau_{zy}\right)+\rho f_y$$

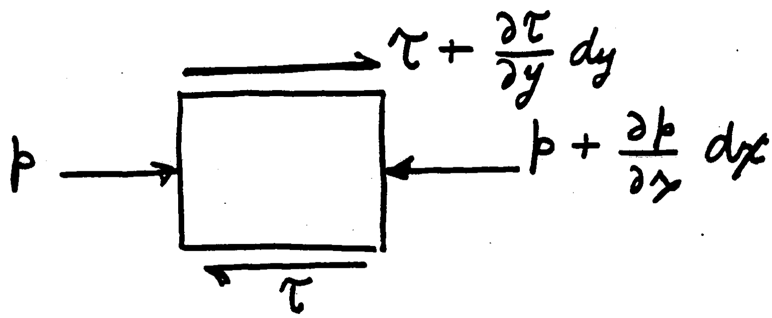
$$\frac{\partial}{\partial t}\rho w=-\frac{\partial p}{\partial z}-\left(\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial w}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xz}+\frac{\partial}{\partial y}\tau_{xz}+\frac{\partial}{\partial z}\tau_{zz}\right)+\rho f_z$$

ENERGY

$$\rho c_v\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)-T\left(\frac{\partial p}{\partial T}\right)_p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$
$$-\left(\tau_{xx}\frac{\partial u}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}\right)-\left(\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\tau_{xz}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\tau_{yz}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)$$

PARALLEL FLOW





$$\Sigma F_x = p\, dy\, dz - \left(p + \frac{\partial p}{\partial x} dx\right) dy\, dz$$

$$+ \tau\, dx\, dy + \left(\tau + \frac{\partial \tau}{\partial y} dy\right) dx\, dz = \rho\, a_x\, d\forall$$

$$\rho \frac{DV_x}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y}$$

$$\tau = \mu \frac{\partial V_x}{\partial y}$$

$$\rho \frac{DV_x}{Dt} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 V_x}{\partial y^2}$$

$$\frac{DV_x}{Dt} = \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = \frac{\partial V_x}{\partial t}$$

VISCOSITY - UNITS

$$\tau = \mu \frac{\partial V}{\partial y}$$

$$\mu = \tau \Big/ \partial V/\partial y$$

ENGLISH

$$\mu \sim \frac{lb/ft^2}{1/sec} = \frac{lb\ sec}{ft^2}$$

$$lb = slug\ ft/sec^2$$

$$\mu = \frac{slug}{ft\ sec}$$

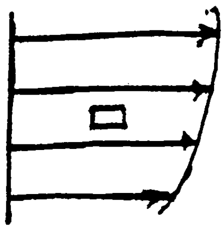
METRIC

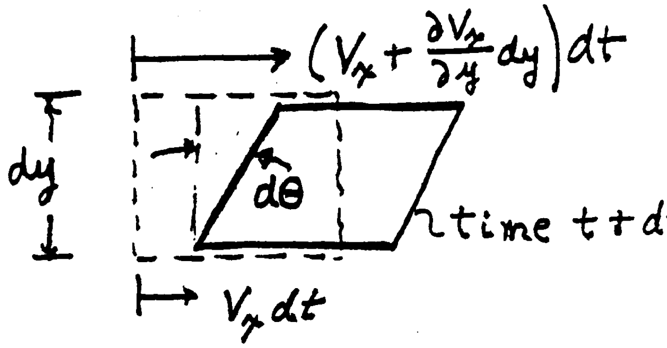
$$\mu \sim \frac{N/m^2}{1/sec} = \frac{N\text{-}sec}{m^2}$$

$$N = Kg\ m/sec^2$$

$$\mu \sim \frac{Kg}{m \cdot sec}$$

STRESS + STRAIN IN A PARALLEL FLOW





$$\tan d\theta \cong d\theta = \frac{\left(V_x + \frac{\partial V_x}{\partial y} dy\right) dt - V_x \, dt}{dy}$$

$$\text{STRAIN} = \frac{d\theta}{dt} = \frac{\partial V_x}{\partial y}$$

STRESS $\propto$ STRAIN RATE (NEWTONIAN FLUID)

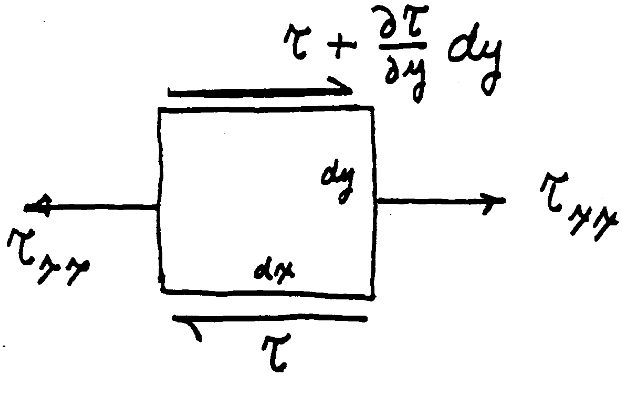
$$\tau_{xy} = \mu \frac{\partial V_x}{\partial y}$$

μ = COEFFICIENT OF VISCOSITY
PROPERTY OF THE FLUID

TABLE 1.2
Properties of common liquids at 1 atm and 20°C

Liquid	Viscosity μ		Kinematic viscosity $\nu = \mu/\rho$	
	kg / (m · s)	slug / (ft · s)	m^2 / s	ft^2 / s
Alcohol (ethyl)	1.2×10^{-3}	2.51×10^{-5}	1.51×10^{-6}	1.62×10^{-5}
Gasoline	2.9×10^{-4}	6.06×10^{-6}	4.27×10^{-7}	4.59×10^{-6}
Mercury	1.5×10^{-3}	3.14×10^{-5}	1.16×10^{-7}	1.25×10^{-6}
Oil (lubricant)	0.26	5.43×10^{-3}	2.79×10^{-4}	3.00×10^{-3}
Water	1.005×10^{-3}	1.67×10^{-5}	0.804×10^{-6}	8.65×10^{-6}

PARALLEL FLOW (STEADY)



FLOW IS DRIVEN BY SHEAR

$$\sum F_x = -\tau \, dx \, dz + \left(\tau + \frac{\partial \tau}{\partial y} dy\right) dx \, dz = \rho \, a_x dV =$$

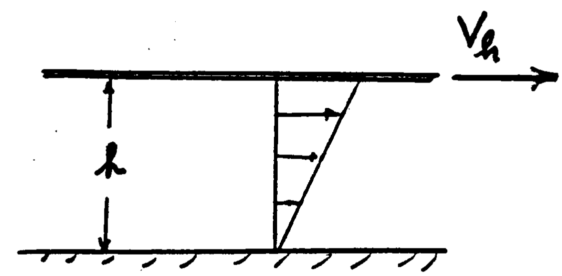
$$\frac{\partial \tau}{\partial y} = 0 \qquad \tau = \text{CONSTANT}$$

$$\tau = \mu \frac{dV}{dy}$$

$$V = \frac{\tau}{\mu} y + C$$

$$V(0) = 0 \qquad C = 0$$

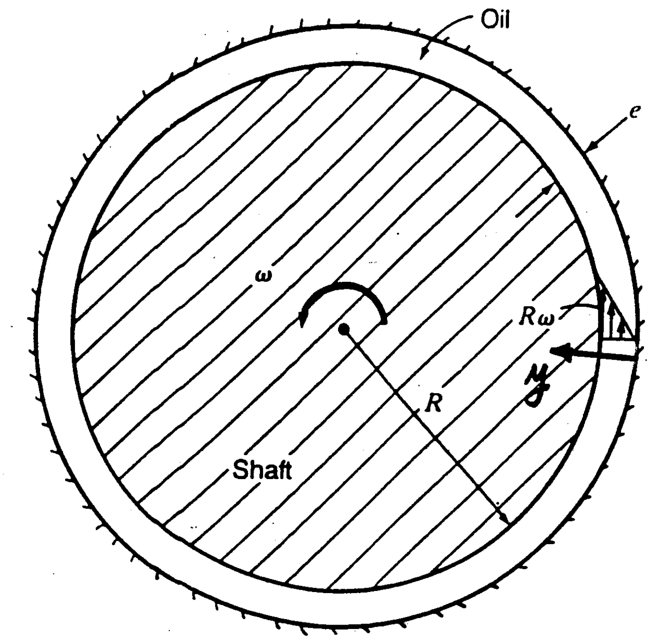
$$V(h) = V_h$$



$$V_h = \frac{\tau}{\mu} h$$

$$\tau = \mu \frac{V_h}{h} \qquad V = V_h \frac{y}{h}$$

ROTATING SHAFT



$$\tau = \mu \frac{\partial V}{\partial y} = \mu \frac{R\omega - 0}{\ell} = \mu \frac{R\omega}{\ell}$$

$$\text{TORQUE} = R(\tau 2\pi R \ell)$$

$R = 3''$

Oil $\mu = 5.43 \times 10^{-3} \ \frac{\text{SLUG}}{\text{ft s}}$ $\ell = 0.1''$

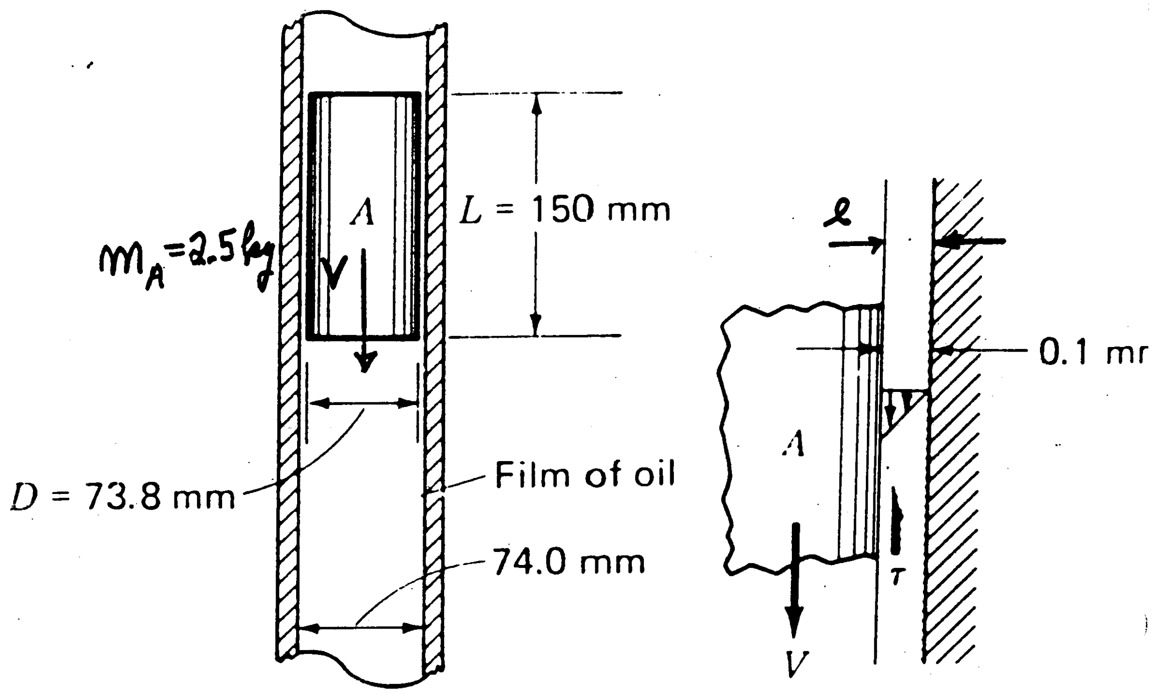
$RPM = 1000 \qquad \omega = 1000\ 2\pi/60 = 104.7 \ \frac{\text{rad}}{\text{s}}$

$$\tau = \frac{5.43 \times 10^{-3}\ (3/12)\ 104.7}{(.1/12)} = 17.06 \ \text{lb/ft}^2$$

$$F = \tau\ 2\pi R \ell = 17.06\ (2\pi)(3/12)(1\text{ft}) = 26.8 \ \text{lb}$$

$$T = RF = 6.7 \ \text{ft lb}$$

EXAMPLE 1.1



What is the terminal velocity of the cylinder?

$$\tau = \mu \frac{\partial V}{\partial y} = \mu \frac{V}{\ell} \qquad \mu = 7 \times 10^{-3} \frac{N \cdot s}{m^2}$$

$$\sum F_y = \tau \pi D L - W = 0$$

$$\mu \frac{V}{\ell} \pi D L = W$$

$$V = \frac{\ell W}{\mu \pi D L} = \frac{.0001\,(2.5)(9.81)}{7 \times 10^{-3}\,\pi\,(.0738)(.150)}$$

$$= 10.07 \text{ m/s}$$