3 Dynamic Behavior of Measurement Systems

Order of a Dynamic Measurement System

Every measurement system responds to inputs in a unique way. For example, your ability to hear high frequency sounds will probably degrade as you age and will never be as keen as most dogs hearing. Sound pressure waves are a dynamic signal and the sensing of these pressure waves by a flexible membrane (like your ear drum) can be mathematically modeled and therefore simulated.

Our goal in this section is to apply our understanding of the physics involved in sensing a signal and build a mathematical model that could be used to describe the response of the measurement system to a dynamic signal. In prior sections we described the response of a measurement system to a static signal and built a mathematical model which described that response. The process of characterizing that response is referred to as a static calibration and the resulting mathematical model is called the static calibration curve.

In the first lab you will perform both a static and dynamic calibration of a temperature sensor and determine the corresponding static and dynamic models which describe the sensor response. In the case of a signal that is changing with time (dynamic) a sensor that can keep up, or is fast enough, is needed to accurately detect the change. In the case of the temperature sensors used in the first lab both the sensor and the environment being sensed must be at the same temperature to make an accurate measurement. If the sensor is initially at a different temperature then some amount of time is required for the sensor and the environment to become the same temperature. There has been a dynamic change in the sensor temperature in response to a dynamic change in the input temperature signal.

In this example we understand that heat must be transferred from the environment to the sensor. The physics of that heat transfer might be modeled based on our understanding of conduction, convection, radiation or possibly some combination thereof. In general we could reason that the temperature sensor performs some mathematical operation on the input signal and outputs the result.

In fact most measurement systems can be modeled using a differential equation that describes the relationship between the input signal and the output signal. In the first lab you will find the linear equation that describes the response to a static input (a static calibration) and the first order differential equation that describes the conductive heat transfer to and from the sensor (a dynamic calibration).

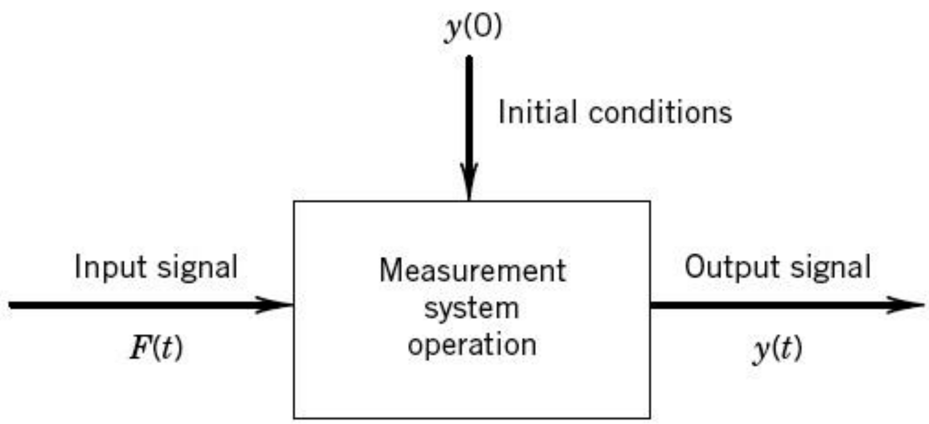


Figure 3.2 Measurement system operation on an input signal, $F(t)$, provides the output signal, $y(t)$.

Measurement System Model

If the measurement system operation performed on the input signal, $F(t)$, in figure 3.2 is an n^{th}-order linear differential equation then the output signal, $y(t)$, can be represented with the equation:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} \ldots + a_1 \frac{dy}{dt} + a_0 y = F(t) \qquad (3.1)$$

where the coefficients, a_0, a_1, a_2, ..., a_n represent the physical system parameters whose properties and values will depend on the measurement system itself. The forcing function, $F(t)$, can also be generalized into an m^{th}-order equation of the form:

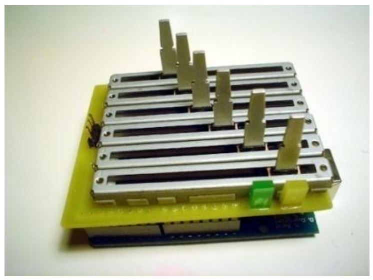
$$F(t) = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} \ldots + b_1 \frac{dx}{dt} + b_0 x \qquad m \le n$$

where b_0, b_1, ..., b_m also represent physical system parameters. The nature of these equations should reflect the governing equations of the pertinent fundamental physical laws of nature that are relevant to the measurement system.

Zero-Order System

If all the derivatives in Equation 3.1 are zero then the most basic model of a measurement system is obtained, the zero-order differential equation:

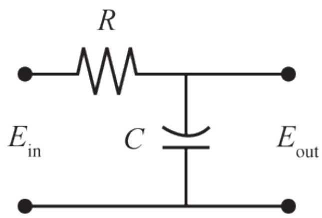
$$a_0 y = F(t)$$



From this equation it is easy to see that any input, $F(t)$, is instantly reflected in the output y with only a factor, a_0, modification. If the input is a dynamically varying signal b_0x then $y = b_0/a_0 x$ or $y = Kx$. The factor K is often times referred to as the *static sensitivity* found during a static calibration.

First-Order System

A linear time-invariant (LTI) first-order system contains a single mode of energy storage. A simple Resister-Capacitor circuit is a first order system.



Here the underlying physics is described by the equation

$$RC\frac{dV_{out}}{dt}+V_{out}=V_{in}$$

This circuit is called a single pole low-pass RC filter and will be discussed in greater detail in subsequent sections on signal conditioning and filters.

Systems with thermal capacity like a bulb thermometer or thermocouple require heat transfer, Q, from their environment to effect a sensor temperature change. The change in energy, E, with respect to time is described by the first-order equation.

$$Q=\frac{dE}{dt}=mC_v\frac{dT}{dt}=hA_s\;\; T_\infty(t)-T_s(t)$$

where m is the sensor's mass, C_v is the sensor's specific heat, h is the convective heat transfer coefficient, A_s is the surface area of the sensor, T_∞ is the temperature of the surrounding material and T_s is the temperature of the sensor. This can be rearranged as

$$mC_v\frac{dT}{dt}=hA_sT_\infty(t)-hA_sT_s(t)$$
$$mC_v\frac{dT}{dt}+hA_sT_s(t)=hA_sF(t)$$

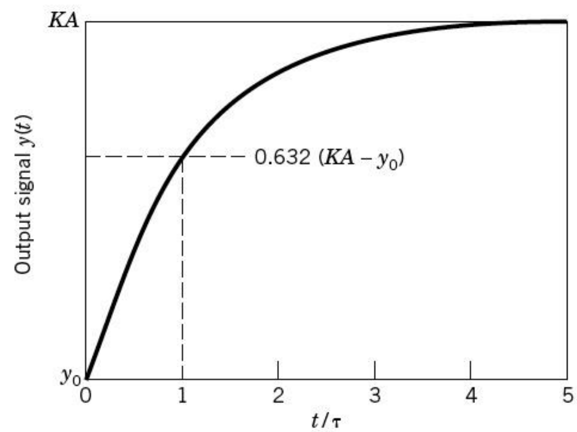
This can obviously be represented as a first-order differential equation in the form of equation 3.1 as

$$a_1\frac{dy}{dt}+a_0y=F(t)$$

To help clarify the underlying physics the equation can be recast by dividing through by a_0 and setting $\dot{y}=dy/dt$.

$$\tau \dot{y} + y = KF(t)$$

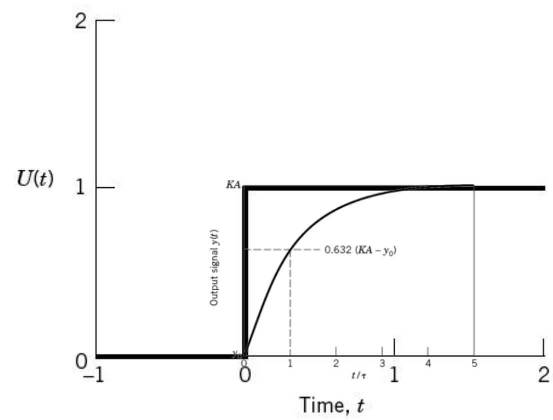
where $\tau = a_1/a_0$. The parameter τ is called the *time constant* of the system. Reflecting back it is easy to see that the time constant of a single-pole low-pass RC filter is $1/RC$ and that of a temperature sensor is based on the mass, specific heat, heat transfer coefficient, and the surface area of the sensor, mC_v/hA_s.



It is essential that you grasp the insight that the time constant of such systems (LTI) or sensors is based on properties that do not change (under normal operating conditions). I.e. a bulb thermometer does not change in mass when subjected to a temperature change nor does its specific heat, surface area or heat transfer coefficient change therefore its time constant ***remains constant.***

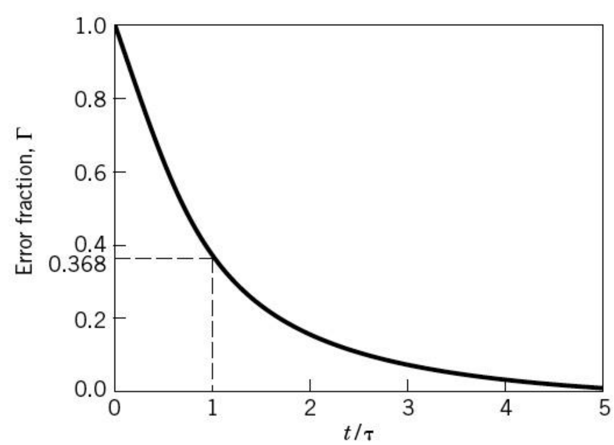
Dynamic Calibration of a First-Order System

Like a static calibration, the sensor is subjected to a known input and the resulting sensor output is recorded. For a dynamic calibration a dynamic input is needed and the ability to measure a time varying signal is required. Being engineers is makes sense that we would start with equations that model the physics and find a simple solution to them. If the input function, F(t), is a unit step function, U(t), then $\tau \dot{y} + y = KF(t)$ can be recast as, $\tau \dot{y} + y = y_\infty$ which has the solution:



$$y = y_\infty + [y_0 - y_\infty]e^{-t/\tau}$$

Recall that the unit step function, $U(t)$, is zero for all time prior to t_0 and 1 for all time thereafter. In practice $U(t)$ usually has an amplitude, A, other than 1. The difference between the input and the output is often referred to as the error. With a simple rearrangement of terms that error is clearly shown to be an exponential function



$$\Gamma = \frac{y - y_\infty}{[y_0 - y_\infty]} = e^{-t/\tau}$$

When $t = \tau$ the error function is $e^{-1} = 0.368$ or $y = 0.632(KA - y_0)$.

By taking the natural log of the error function or when plotted in semi-log coordinates the equation assumes a linear form.

$$\ln \Gamma = \ln\left[\frac{y - y_\infty}{[y_0 - y_\infty]}\right] = -t/\tau$$

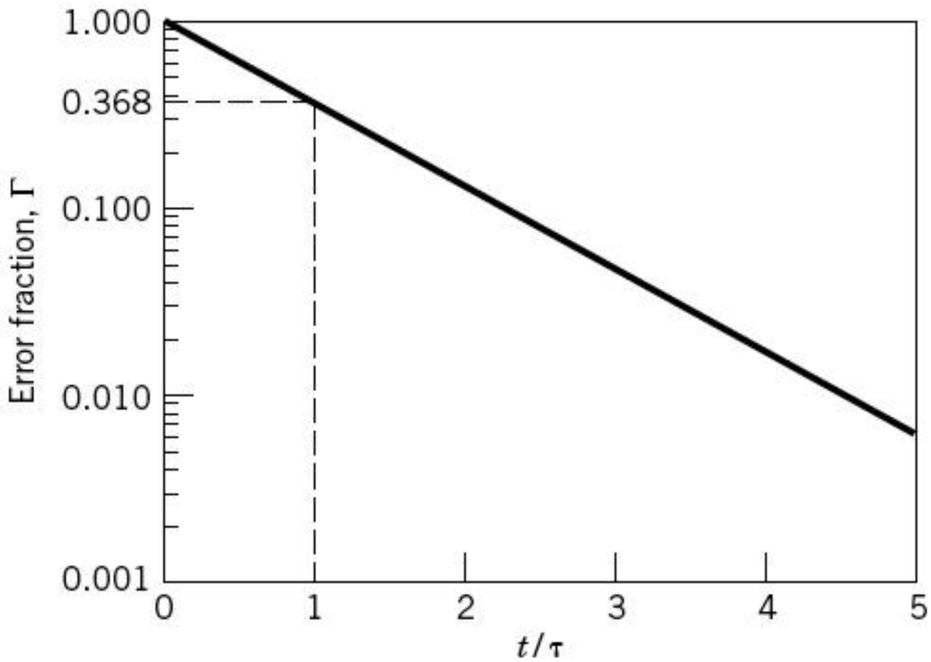


Figure 3.8. The error function plotted in semi-log coordinates.

The slope of the linearized error function is $-1/\tau$. Finding the slope of a line is less sensitive to errors than finding a point on a curve (at a value of $y = 0.632(KA - y_0)$.)

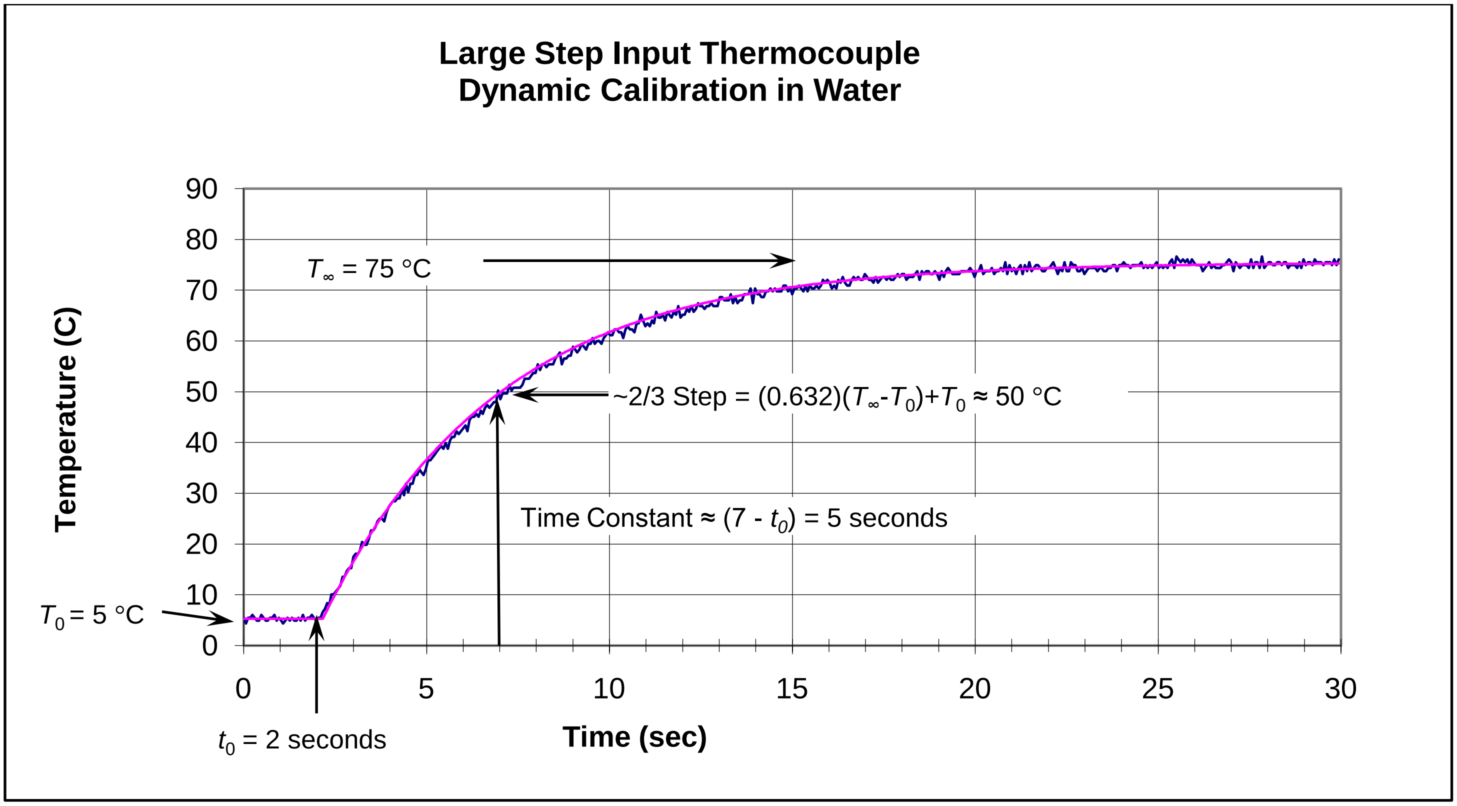
Dynamic Calibration of Thermocouple

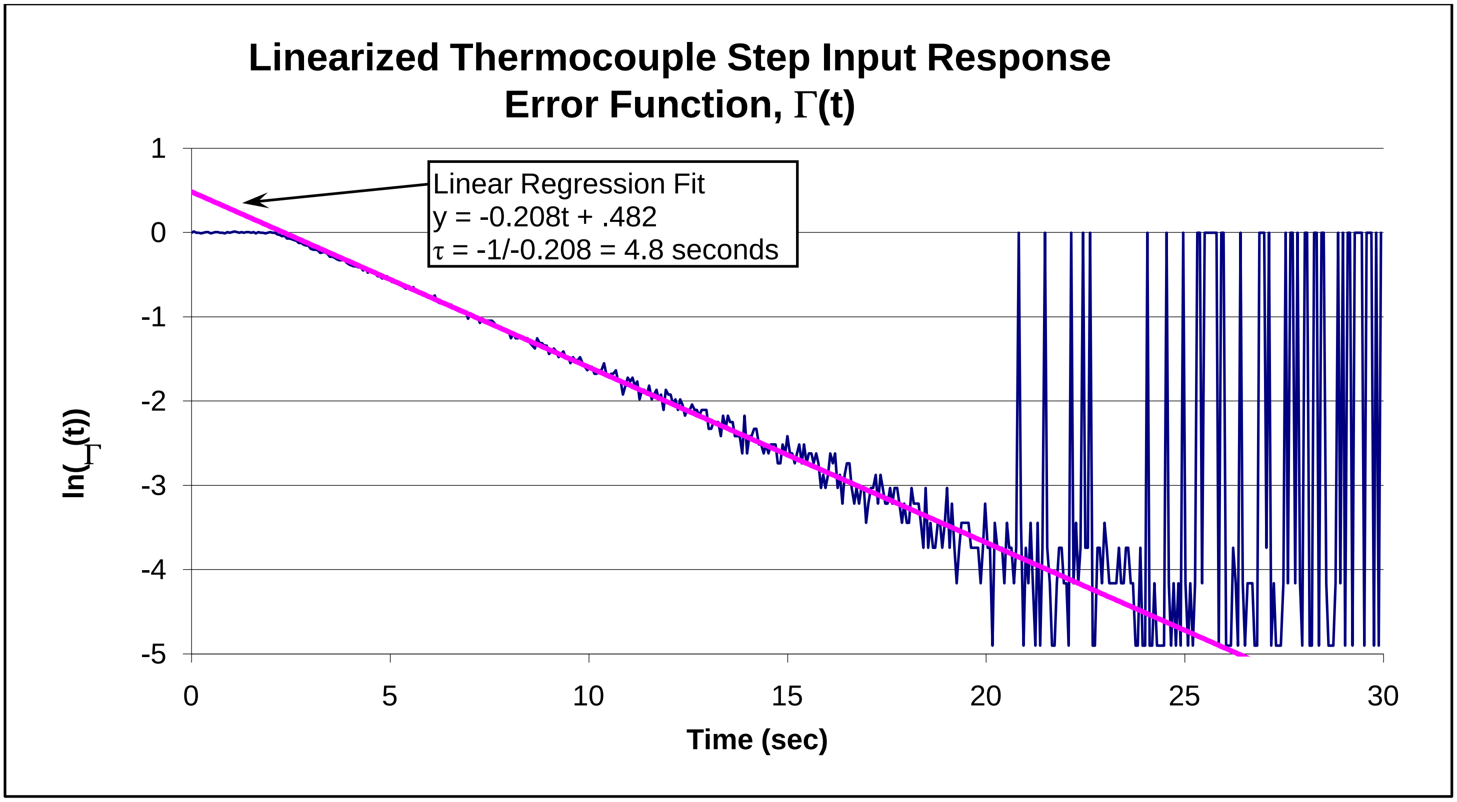
In the first lab you will be performing a dynamic calibration of a thermocouple by subjecting it to a sudden change in temperature (i.e. moving it from cold water to warm water).

The governing equation from above $mC_v \frac{dT}{dt} + hA_s T_s(t) = hA_s F(t)$ can easily be recast in the more familiar form

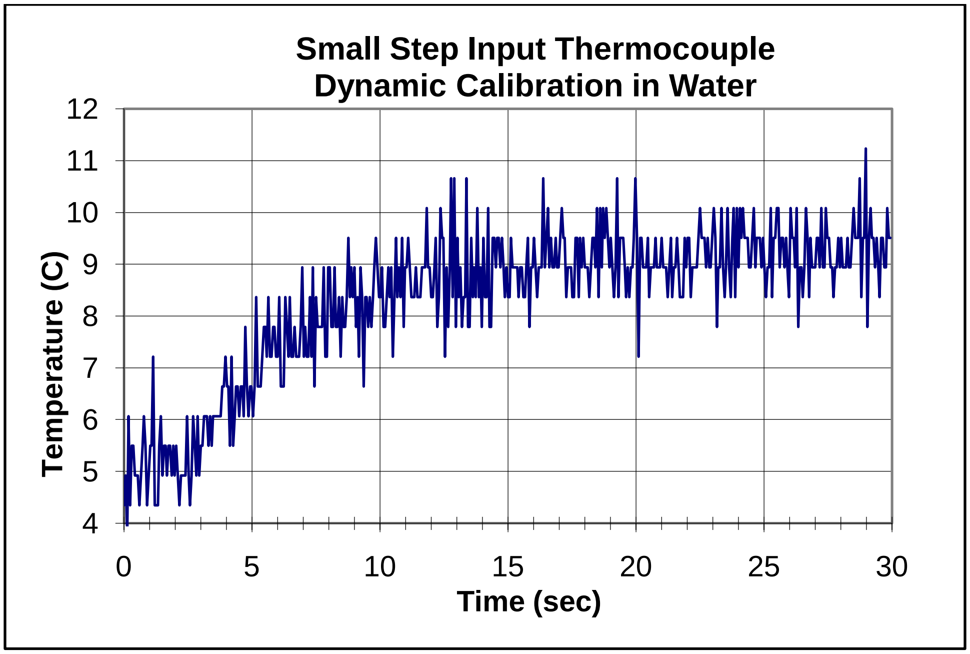
$$\frac{mC_v}{hA_s} \frac{dT}{dt} + T_s(t) = F(t)$$

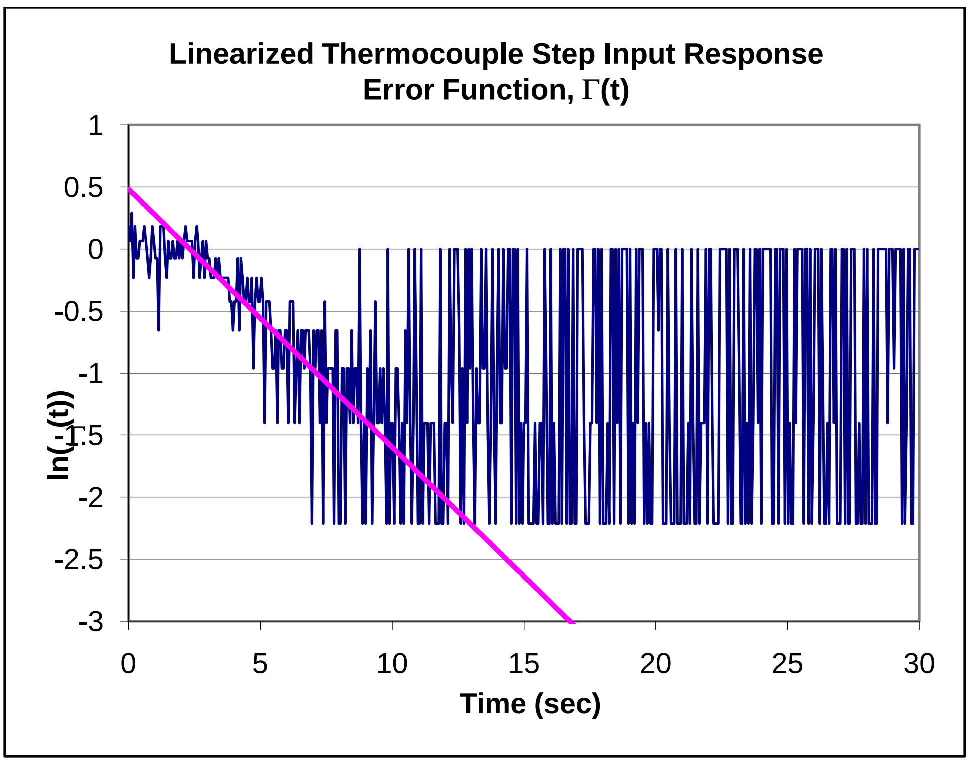
where the time constant is defined by the physical constants $\frac{mC_v}{hA_s}$. Here the natural log of the error function is plotted in linear coordinates and a line is fit to a portion of the data from 5 to 15 seconds. The slope of that line, -0.208, is -1/time constant or τ = 4.8 seconds.


Large Step Input Thermocouple
Dynamic Calibration in Water
T∞ = 75 °C
~2/3 Step = (0.632)(T∞-T0)+T0 ≈ 50 °C
Time Constant ≈ (7 - t0) = 5 seconds
T0 = 5 °C
t0 = 2 seconds
Temperature (C)
Time (sec)
0
10
20
30
40
50
60
70
80
90
0
5
10
15
20
25
30


Linearized Thermocouple Step Input Response
Error Function, Γ(t)
Linear Regression Fit
y = -0.208t + .482
τ = -1/-0.208 = 4.8 seconds
ln(Γ(t))
Time (sec)
1
0
-1
-2
-3
-4
-5
0
5
10
15
20
25
30

An example data set using the same thermocouple containing considerable noise and quantization error is plotted below. Of note is the difficulty with which ~2/3 of the step response could be determined. In contrast the time constant determined from the linearized error function is relatively insensitive to the noise and quantization errors.





The line above is fit to only the first 2 seconds (~3 to 5 seconds) of the linearized error function of the step input response and yields a τ = 4.8 s.

Frequency Response of a First-Order System

The determination of the static sensitivity and time constant of a first-order system transforms the black box "Measurement system operation" in figure 3.2 into a known function. That implies that for any given output which is correctly recorded the input that produced it could be ascertained. This is possible because the differential equation describing the physics of the measurement system is known and is solvable with relative ease.

An intuitive description of the relationship between a system input and output requires an understanding of the user, their intent as well as the measurement system. Most users of thermometers are not interested in dynamic temperature measurement. It is fair to say that few have the training needed to relate the math to the physics and then apply this knowledge to understand a dynamic phenomenon or solve a problem.

To facilitate an understanding of a system's dynamic response we will start by finding a solution to a simple sine wave input.

$$\tau \frac{dy}{dt} + y = KA\sin(t)$$

We already know that the complementary equation, $\tau \, {}^{dy}/_{dt} + y = 0$ has a decaying exponential solution of the form $y(t) = Ce^{-t/\tau}$. By applying appropriate initial conditions a particular solution at a single frequency, ω_1, can be found in the form of $y(t) = B_1 \sin[\omega_1 t + \phi(\omega_1)]$.

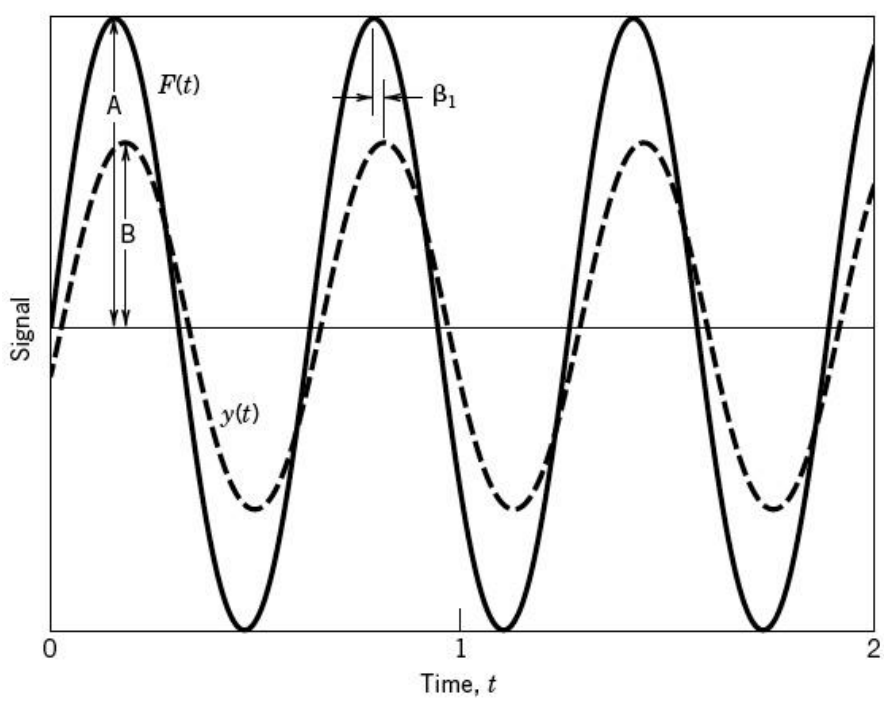


Figure 3.11 Relationship between a sinusoidal input and output: amplitude, frequency, and time lag.

For an input $A_1 \sin(\omega_1 t)$ the output $B_1 \sin[\omega_1 t + \phi(\omega_1)]$ is produced. There is an amplitude reduction from A_1 to B_1 and a delay in time, β_1, reflected in the phase shift of $\phi(\omega_1)$.

The complete solution is

$$y(t) = B(\omega)\sin[\omega t + \phi(\omega)] + Ce^{-t/\tau}$$

where $B(\omega) = KA/\left[1+(\omega\tau)^2\right]^{1/2}$ and $\phi(\omega) = \tan^{-1}(\omega\tau)$. This solution depends only on the static sensitivity, K, and the time constant, τ. The time constant is the only system characteristic that affects the frequency response. This solution provides a relationship between the input and output for all frequencies. The ratio of output/input magnitude would therefore be $M(\omega) = B(\omega)/KA$. Figure 3.12 is plot of the magnitude ratio versus the normalized frequency $\tau\omega$. Note that at $\tau\omega = 1$ the magnitude is 0.707 which can be derived from $M(\omega) = 1/\left[1+(\omega\tau)^2\right]^{1/2} = 1/\sqrt{2}$.

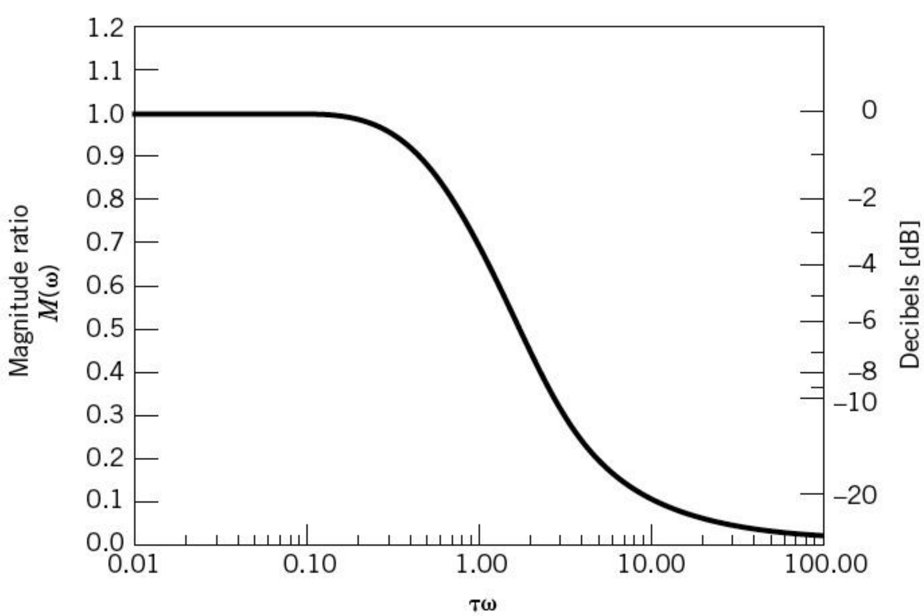
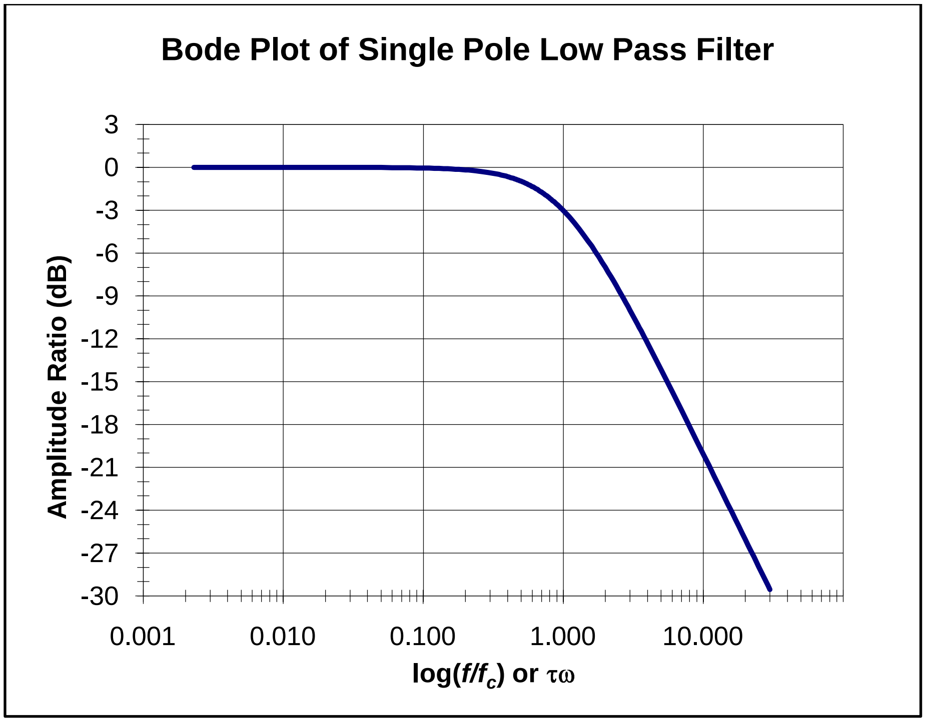


Figure 3.12 First-order system frequency response: magnitude ratio.

Figure 3.12 is plotted with the magnitude ratio expressed in decibels below. A decibel, dB, is unit of power defined as 20 $\log_{10}(M(\omega))$.

The **frequency response** of a first-order sensor is defined based on the time constant as $\omega = 1/\tau$. This can be stated in terms of the magnitude ratio as the -3 dB point or the point when the magnitude ratio is 0.707. This definition also carries over to other non-first-order systems even though they are based on a different mathematical model (physics).



A first-order system can be thought of as a low pass filter. They attenuate higher frequencies and pass lower frequencies with little attenuation.

A first-order system always delays the input signal in time. That delay results in a phase shift as evidenced by the $\phi(\omega)$ term in the complete solution. The time delay, β_1, in figure 3.11 above can be solved for as

$$\beta_1 = \frac{\phi(\omega_1)}{\omega_1} = \frac{\tan^{-1}(\omega_1 \tau)}{\omega_1}$$

By removing the particular frequency, ω_1, from the above equation a plot similar to the magnitude ratio can be generated for all frequencies.

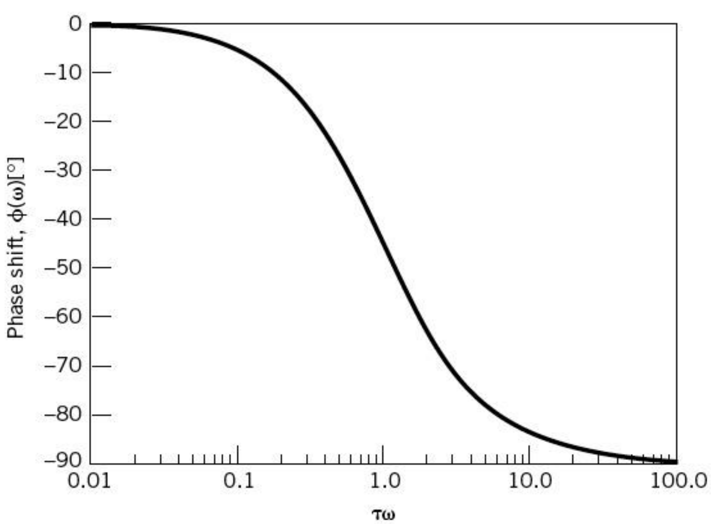


Figure 3.13 First-order system frequency response: phase shift.

Example

Suppose I want to measure a temperature which fluctuates with a frequency of 0.1 Hz with a minimum of 98% amplitude reduction.

ASSUMPTIONS: basic first-order temperature sensor like a thermocouple

FIND: Magnitude ratio of at least 0.98

$$M(\omega) \geq 0.98, \text{ or dB} = 20\log 0.98 = -0.175$$

$$M(\omega) = \frac{B}{KA} = \frac{1}{\left[1+(\omega\tau)^2\right]^{1/2}}$$

rearranging gives

$$\omega\,\tau = \left[1/M(\omega)^2 - 1\right]^{1/2}$$

$$\text{so for } M(\omega) \geq 98\%, \ \ \omega\tau \leq 0.2$$

$$\text{or, } \tau \leq 0.2/\omega = 0.2/2\pi f = 0.2/\ \ 2\times 3.142\times 0.1$$

$$\tau \leq 0.31\,\text{sec}$$

Problem 3.7

A thermocouple has a time constant of 20 ms. Determine its 90% rise time.

$$\mathbf{90\% = 1 - \Gamma(t) = 1 - e^{-t/\tau}}$$

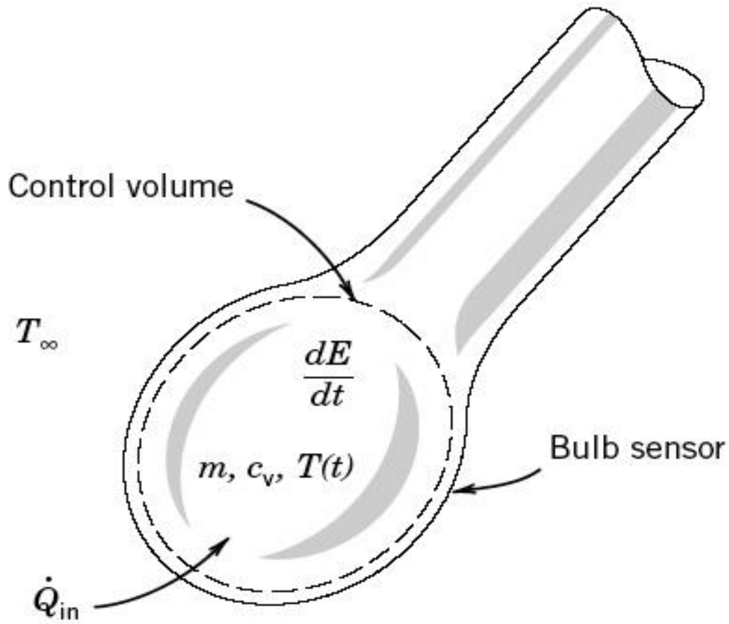
$$\mathbf{.9 - 1 = -.1 = -e^{-t/\tau}}$$

$$\boldsymbol{ln(0.1) = -t/\tau}$$

$$\boldsymbol{t_{90} = 2.3\tau = 46\ ms}$$

Example 3.3

Suppose a bulb thermometer originally indicating 20ºC is suddenly exposed to a fluid temperature of 37 ºC. Develop a simple model to simulate the thermometer output response.



KNOWN:

T_0 = 20ºC

T_∞ = 37ºC

F(t) = [T_∞ - T_0]U(t)

ASSUMPTIONS:

Normal first-order response

FIND: T(t)

SOLUTION:

The rate at which energy is exchanged between the sensor and the environment through convection, $\dot{\boldsymbol{Q}}$, must be balanced by the storage of energy within the thermometer, *dE/dt*.

$$\frac{dE}{dt} = \dot{Q}$$

For a constant mass temperature sensor,

$$\frac{dE(t)}{dt} = mc_v\frac{dT(t)}{dt} = hA\Delta T = hA_s[T_\infty - T(t)]$$

This can be written in the form

$$mc_v\frac{dT(t)}{dt} + hA_s[T(t) - T(0)] = hA_s[T_\infty - T(0)]U(t)$$

dividing by hA_s

$$\frac{mc_v}{hA_s}\frac{dT(t)}{dt} + T(t) = T_\infty$$

Therefore:

$\tau = \frac{mc_v}{hA_s}$, $K = \frac{hA_s}{hA_s} = 1$

The thermometer response is therefore:

$$T(t) = T_\infty + [T(0) - T_\infty]e^{-t/\tau}$$

$$= 37 - 17e^{-t/\tau}\ [^\circ\mathrm{C}]$$