A Systematic Design of High-Rate Complex Orthogonal Space-Time Block Codes

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Abstract—In this letter, a systematic design method to generate high-rate space-time block codes from complex orthogonal designs for any number of transmit antennas is proposed. The resulting designs have the best known rates. Two constructions with rates $2/3$ and $5/8$ are further illustrated for 6 and 7 transmit antennas, respectively.

Index Terms—Complex orthogonal designs, diversity, space-time block codes (STBCs).

I. INTRODUCTION

SPACE-TIME BLOCK CODES (STBCs) from complex orthogonal designs have received extensive interest recently, see for example [1]–[9]. The special structure of orthogonal designs guarantees that these codes achieve full diversity and have a simple maximum-likelihood decoding algorithm.

A *complex orthogonal design* (COD) in variables $x_1, x_2, \ldots, x_k$ is a $p \times n$ matrix G such that: 1) the entries of G are complex linear combinations of $x_1, x_2, \ldots, x_k$ and their complex conjugates $x_1^*, x_2^*, \ldots, x_k^*$ and 2) the columns of G are orthogonal to each other, i.e.,

$$G^{\mathcal{H}}G = (|x_1|^2 + |x_2|^2 + \cdots + |x_k|^2)I_n, \qquad (1)$$

where I_n is the $n \times n$ identity matrix, and the superscript $\mathcal{H}$ stands for the complex conjugate and transpose of a matrix. The rate of G is defined as $R = k/p$. This design G can be used to generate a STBC for n transmit antennas. The resulting STBC has a rate k/p, which means that each codeword with block length p carries k symbols.

The first STBC from COD was proposed in Alamouti [1] for two transmit antennas, which is the following 2×2 design:

$$G_2 = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}. \qquad (2)$$

Clearly, the rate of G_2 is 1. For $n = 4$ transmit antennas, there are CODs of rate $R = 3/4$ [2]–[4], for example,

$$G_4 = \begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2^* & x_1^* & 0 & x_3 \\ -x_3^* & 0 & x_1^* & -x_2 \\ 0 & -x_3^* & x_2^* & x_1 \end{bmatrix}. \qquad (3)$$

The design for $n = 3$ transmit antennas is obtained simply by taking the first three columns of G_4. A class of CODs with rate $R = 1/2$ was given by Tarokh *et al.* in [2] by taking advantage of the full-rate *real* orthogonal designs [2], [10]. Later in [5], two generalized CODs with rates $7/11$ and $3/5$ were constructed for $n = 5$ and $n = 6$ transmit antennas, respectively. Recently, a COD with rate $2/3$ was presented for $n = 5$ transmit antennas [7].

In case of CODs of square size ($p = n$), it has been shown in [2], [3] that 4×4 CODs of rate 1 do not exist, and a general result was given in [4] that $R \leq (a + 1)/(2^a b)$ if $n = 2^a b, b$ odd, which is related to the Hurwitz theory [11], [12]. In case of CODs of nonsquare size ($p \geq n$), it was proved in [8] that rate 1 cannot be achieved for $n \geq 3$ transmit antennas. Later in [6], we showed that for CODs without linear processing, the rate cannot be greater than $3/4$ for $n \geq 3$ transmit antennas. Further in [9], it was showed that this result holds for any COD.

In this letter, we first present two hand-crafted high-rate CODs for $n = 6$ and $n = 7$ transmit antennas explicitly. Then we propose a general algorithm to generate high-rate CODs for any number of transmit antennas.

II. CODs FOR ANY NUMBER OF TRANSMIT ANTENNAS

For $n = 6$ transmit antennas, we construct a 30×6 COD with rate $R = 2/3$ as shown in matrix G_6, at the left-hand column of the next page. This design contains $k = 20$ symbols and has a block length or delay $p = 30$. For $n = 7$ transmit antennas, a 56×7 COD G_7 is specified as shown in the matrix, shown at the right-hand column of the next page. This design contains $k = 35$ symbols and has a block length $p = 56$. Clearly, the rate of G_7 is $R = k/p = 5/8$.

The basic methodology of constructing the orthogonal designs G_6 and G_7 is stated as follows. For convenience of explanation, let us say a row in an orthogonal design is *conjugate (nonconjugate)* if all symbols except zeros in this row have (do not have) complex conjugate. For example, the first row of the orthogonal design G_2 in (2) is nonconjugate, and the second row is conjugate. We started from $G_1 = x_1 I_1$, and generated G_m from G_{m-1} iteratively for any $2 \leq m \leq n$. In each iteration from G_{m-1} to G_m, we added a new column with some new symbols. The number of the new symbols and the position of each new symbol in the new column depend on the numbers of conjugate rows and nonconjugate rows in G_{m-1}. If the number of nonconjugate rows is not less than that of conjugate rows, we add some new symbols into the nonconjugate rows and set some zeros in the conjugate rows. If the number of nonconjugate rows is less than that of conjugate rows, we put some new symbols with complex conjugate into the conjugate rows

$$
G_6 = \begin{bmatrix}
x_1 & x_2 & x_3 & 0 & x_7 & 0 \\
-x_2^* & x_1^* & 0 & x_4^* & 0 & x_{11}^* \\
-x_3^* & 0 & x_1^* & x_5^* & 0 & x_{12}^* \\
0 & -x_3^* & x_2^* & x_6^* & 0 & x_{13}^* \\
0 & -x_4 & -x_5 & x_1 & x_8 & 0 \\
x_4 & 0 & -x_6 & x_2 & x_9 & 0 \\
x_5 & x_6 & 0 & x_3 & x_{10} & 0 \\
-x_6^* & x_5^* & -x_4^* & 0 & 0 & x_{14}^* \\
-x_7^* & 0 & 0 & -x_8^* & x_1^* & x_{15}^* \\
0 & -x_7^* & 0 & -x_9^* & x_2^* & x_{16}^* \\
0 & 0 & -x_7^* & -x_{10}^* & x_3^* & x_{17}^* \\
-x_9^* & x_8^* & 0 & 0 & x_4^* & x_{18}^* \\
-x_{10}^* & 0 & x_8^* & 0 & x_5^* & x_{19}^* \\
0 & -x_{10}^* & x_9^* & 0 & x_6^* & x_{20}^* \\
x_8 & x_9 & x_{10} & -x_7 & 0 & 0 \\
0 & -x_{11} & -x_{12} & 0 & -x_{15} & x_1 \\
x_{11} & 0 & -x_{13} & 0 & -x_{16} & x_2 \\
x_{12} & x_{13} & 0 & 0 & -x_{17} & x_3 \\
0 & 0 & x_{14} & -x_{11} & -x_{18} & x_4 \\
0 & -x_{14} & 0 & -x_{12} & -x_{19} & x_5 \\
x_{14} & 0 & 0 & -x_{13} & -x_{20} & x_6 \\
x_{15} & x_{16} & x_{17} & 0 & 0 & x_7 \\
0 & -x_{18} & -x_{19} & x_{15} & 0 & x_8 \\
x_{18} & 0 & -x_{20} & x_{16} & 0 & x_9 \\
x_{19} & x_{20} & 0 & x_{17} & 0 & x_{10} \\
-x_{13}^* & x_{12}^* & -x_{11}^* & -x_{14}^* & 0 & 0 \\
-x_{16}^* & x_{15}^* & 0 & x_{18}^* & -x_{11}^* & 0 \\
-x_{17}^* & 0 & x_{15}^* & x_{19}^* & -x_{12}^* & 0 \\
0 & -x_{17}^* & x_{16}^* & x_{20}^* & -x_{13}^* & 0 \\
x_{20}^* & -x_{19}^* & x_{18}^* & 0 & x_{14}^* & 0
\end{bmatrix}
$$

and set some zeros in the nonconjugate rows accordingly. Due to the new symbols in the mth column, we have to arrange some additional rows to guarantee the orthogonality.

With the basic ideas, we have developed a general algorithm as follows. Denote k_{m-1} and p_{m-1}, respectively, the numbers of symbols and rows in G_{m-1} for any $m \geq 2$.

- *Initialization:* Start from $G_1 = x_1 I_1$.
- *Generate G_m from G_{m-1}:* for $m = 2, 3, \ldots, n$
  1) Calculate ν_0 and ν_1, the numbers of nonconjugate rows and conjugate rows in G_{m-1}, respectively.
  2) Add some new symbols into the mth column from the first row to the p_{m-1}th row. If $\nu_0 \geq \nu_1$, then add new symbols $x_{k_{m-1}+1}, \ldots, x_{k_{m-1}+\nu_0}$ into the nonconjugate rows, and set zeros into the conjugate rows; If $\nu_0 < \nu_1$, then add new symbols with complex conjugate $x_{k_{m-1}+1}^*, \ldots, x_{k_{m-1}+\nu_1}^*$ into the conjugate rows, and set zeros into the nonconjugate rows.
  3) Since the symbols $x_1, x_2, \ldots, x_{k_{m-1}}$ must appear in each column, we further specify the entries in the mth column after the p_{m-1}th row as follows. If $\nu_0 \geq \nu_1$ (or $\nu_0 < \nu_1$), then put $x_1^*, x_2^*, \ldots, x_{k_{m-1}}^*$ (or $x_1, x_2, \ldots, x_{k_{m-1}}$) into the mth column from the $(p_{m-1}+1)$th row to the $(p_{m-1}+k_{m-1})$th row.
  4) According to the orthogonality of the mth column to each of the first $m-1$ columns, we fill the entries

$$
\begin{bmatrix}
x_1 & x_2 & x_3 & 0 & x_7 & 0 & x_{21} \\
-x_2^* & x_1^* & 0 & x_4^* & 0 & x_{11}^* & 0 \\
-x_3^* & 0 & x_1^* & x_5^* & 0 & x_{12}^* & 0 \\
0 & -x_3^* & x_2^* & x_6^* & 0 & x_{13}^* & 0 \\
0 & -x_4 & -x_5 & x_1 & x_8 & 0 & x_{22} \\
x_4 & 0 & -x_6 & x_2 & x_9 & 0 & x_{23} \\
x_5 & x_6 & 0 & x_3 & x_{10} & 0 & x_{24} \\
-x_6^* & x_5^* & -x_4^* & 0 & 0 & x_{14}^* & 0 \\
-x_7^* & 0 & 0 & -x_8^* & x_1^* & x_{15}^* & 0 \\
0 & -x_7^* & 0 & -x_9^* & x_2^* & x_{16}^* & 0 \\
0 & 0 & -x_7^* & -x_{10}^* & x_3^* & x_{17}^* & 0 \\
-x_9^* & x_8^* & 0 & 0 & x_4^* & x_{18}^* & 0 \\
-x_{10}^* & 0 & x_8^* & 0 & x_5^* & x_{19}^* & 0 \\
0 & -x_{10}^* & x_9^* & 0 & x_6^* & x_{20}^* & 0 \\
x_8 & x_9 & x_{10} & -x_7 & 0 & 0 & x_{25} \\
0 & -x_{11} & -x_{12} & 0 & -x_{15} & x_1 & x_{26} \\
x_{11} & 0 & -x_{13} & 0 & -x_{16} & x_2 & x_{27} \\
x_{12} & x_{13} & 0 & 0 & -x_{17} & x_3 & x_{28} \\
0 & 0 & x_{14} & -x_{11} & -x_{18} & x_4 & x_{29} \\
0 & -x_{14} & 0 & -x_{12} & -x_{19} & x_5 & x_{30} \\
x_{14} & 0 & 0 & -x_{13} & -x_{20} & x_6 & x_{31} \\
x_{15} & x_{16} & x_{17} & 0 & 0 & x_7 & x_{32} \\
0 & -x_{18} & -x_{19} & x_{15} & 0 & x_8 & x_{33} \\
x_{18} & 0 & -x_{20} & x_{16} & 0 & x_9 & x_{34} \\
x_{19} & x_{20} & 0 & x_{17} & 0 & x_{10} & x_{35} \\
-x_{13}^* & x_{12}^* & -x_{11}^* & -x_{14}^* & 0 & 0 & 0 \\
-x_{16}^* & x_{15}^* & 0 & x_{18}^* & -x_{11}^* & 0 & 0 \\
-x_{17}^* & 0 & x_{15}^* & x_{19}^* & -x_{12}^* & 0 & 0 \\
0 & -x_{17}^* & x_{16}^* & x_{20}^* & -x_{13}^* & 0 & 0 \\
x_{20}^* & -x_{19}^* & x_{18}^* & 0 & x_{14}^* & 0 & 0 \\
-x_{21}^* & 0 & 0 & -x_{22}^* & 0 & -x_{26}^* & x_1^* \\
0 & -x_{21}^* & 0 & -x_{23}^* & 0 & -x_{27}^* & x_2^* \\
0 & 0 & -x_{21}^* & -x_{24}^* & 0 & -x_{28}^* & x_3^* \\
-x_{23}^* & x_{22}^* & 0 & 0 & 0 & -x_{29}^* & x_4^* \\
-x_{24}^* & 0 & x_{22}^* & 0 & 0 & -x_{30}^* & x_5^* \\
0 & -x_{24}^* & x_{23}^* & 0 & 0 & -x_{31}^* & x_6^* \\
0 & 0 & 0 & x_{25}^* & -x_{21}^* & -x_{32}^* & x_7^* \\
-x_{25}^* & 0 & 0 & 0 & -x_{22}^* & -x_{33}^* & x_8^* \\
0 & -x_{25}^* & 0 & 0 & -x_{23}^* & -x_{34}^* & x_9^* \\
0 & 0 & -x_{25}^* & 0 & -x_{24}^* & -x_{35}^* & x_{10}^* \\
-x_{27}^* & x_{26}^* & 0 & x_{29}^* & 0 & 0 & x_{11}^* \\
-x_{28}^* & 0 & x_{26}^* & x_{30}^* & 0 & 0 & x_{12}^* \\
0 & -x_{28}^* & x_{27}^* & x_{31}^* & 0 & 0 & x_{13}^* \\
-x_{31}^* & x_{30}^* & -x_{29}^* & 0 & 0 & 0 & x_{14}^* \\
-x_{32}^* & 0 & 0 & -x_{33}^* & x_{26}^* & 0 & x_{15}^* \\
0 & -x_{32}^* & 0 & -x_{34}^* & x_{27}^* & 0 & x_{16}^* \\
0 & 0 & -x_{32}^* & -x_{35}^* & x_{28}^* & 0 & x_{17}^* \\
-x_{34}^* & x_{33}^* & 0 & 0 & x_{29}^* & 0 & x_{18}^* \\
-x_{35}^* & 0 & x_{33}^* & 0 & x_{30}^* & 0 & x_{19}^* \\
0 & -x_{35}^* & x_{34}^* & 0 & x_{31}^* & 0 & x_{20}^* \\
x_{22} & x_{23} & x_{24} & -x_{21} & -x_{25} & 0 & 0 \\
x_{26} & x_{27} & x_{28} & 0 & x_{32} & -x_{21} & 0 \\
0 & -x_{29} & -x_{30} & x_{26} & x_{33} & -x_{22} & 0 \\
x_{29} & 0 & -x_{31} & x_{27} & x_{34} & -x_{23} & 0 \\
x_{30} & x_{31} & 0 & x_{28} & x_{35} & -x_{24} & 0 \\
-x_{33} & -x_{34} & -x_{35} & x_{32} & 0 & x_{25} & 0
\end{bmatrix}
$$

of the first $m-1$ columns from the $(p_{m-1}+1)$th row to the $(p_{m-1}+k_{m-1})$th row as follows.

```
for row = 1 : p_{m-1}; for col = 1 : m - 1
  if |G_{m-1}(row, col)| = |x_i| for any 1 ≤ i ≤ m - 1,
    if sign(G_{m-1}(row, col)) = 1, then
      G_m(p_{m-1} + i, col) = -G_m^*(row, m);
    if sign(G_{m-1}(row, col)) = -1, then
      G_m(p_{m-1} + i, col) = G_m^*(row, m);
end;
```

Set zeros to those unspecified entries;

5) After Step 4, the mth column is ready. However, we need to arrange some additional rows to ensure the orthogonality of the first $m - 1$ columns. In the additional rows, simply put zeros at the mth column.

Set RowCount $= p_{m-1} + k_{m-1}$.

```
for row = p_{m-1} + 1 : p_{m-1} + k_{m-1}
  if G_m(row, col1) ≠ 0 and G_m(row, col2) ≠ 0 for
1 ≤ col1 < col2 ≤ m - 1, then
    Set NeedNewRow = TRUE;
    X_11 = G_m(row, col1); X_12 = G_m(row, col2);
    for NewRow = p_{m-1} + k_{m-1} + 1 : RowCount
      X_21 = G_m(NewRow, col1);
      X_22 = G_m(NewRow, col2);
      if X_21 = 0 and |X_22| = |X_11|, then
        if X_11 = X_22^*, then
          G_m(NewRow, col1) = -X_12^*;
        if X_11 = -X_22^*, then
          G_m(NewRow, col1) = X_12^*;
        NeedNewRow = FALSE;
      if |X_21| = |X_12| and X_22 = 0, then
        if X_21 = X_12^*, then
          G_m(NewRow, col2) = -X_11^*;
        if X_21 = -X_12^*, then
          G_m(NewRow, col1) = X_11^*;
        NeedNewRow = FALSE;
      if |X_21| = |X_12| and |X_22| = |X_11|, then
        NeedNewRow = FALSE;
    end;
    if NeedNewRow = TRUE, then
      RowCount = RowCount + 1;
      G_m(RowCount, col1) = -G_m^*(row, col2);
      G_m(RowCount, col2) = G_m^*(row, col1);
end;
```

- Let $k_m = k_{m-1} + \max(\nu_0, \nu_1)$, $p_m = \text{RowCount}$, and repeat the above iteration if $m < n$.

By Steps 4 and 5, the orthogonality of the resulting design is guaranteed. This algorithm can be used to generate CODs for any number of transmit antennas. In Table I, we list the number of symbols, block length and rate of CODs for $2 \leq n \leq 18$ transmit antennas. We observe from the table that for $n \leq 18$, the rate of the obtained design satisfies a rule of $(n_0 + 1)/(2n_0)$ if $n = 2n_0$ or $n = 2n_0 - 1$, which is conjectured *optimal* in [9] for any n. We believe that the proposed algorithm will continue to generate CODs with the *optimal* rate for $n > 18$.

III. CONCLUSION AND DISCUSSION

We presented in this letter a systematic design method to generate high-rate complex orthogonal STBCs for any number of

TABLE I
HIGH-RATE STBCs FROM COMPLEX ORTHOGONAL
DESIGNS FOR $2 \leq n \leq 18$ TRANSMIT ANTENNAS

	Symbols k	Block Length p	Rate k/p
$n = 2$	2	2	1
$n = 3$	3	4	3/4
$n = 4$	6	8	3/4
$n = 5$	10	15	2/3
$n = 6$	20	30	2/3
$n = 7$	35	56	5/8
$n = 8$	70	112	5/8
$n = 9$	126	210	3/5
$n = 10$	252	420	3/5
$n = 11$	462	792	7/12
$n = 12$	924	1584	7/12
$n = 13$	1716	3003	4/7
$n = 14$	3432	6006	4/7
$n = 15$	6435	11440	9/16
$n = 16$	12870	22880	9/16
$n = 17$	24310	43758	5/9
$n = 18$	48620	87516	5/9

transmit antennas. Also, the designs for $n \leq 18$ transmit antennas were obtained and two designs with rates $2/3$ and $5/8$ were further illustrated for 6 and 7 transmit antennas, respectively. All of the resulting CODs have the best known rates that are conjectured optimal. Moreover, these designs indicate that CODs with nonsquare size do provide larger rates than those with square size. For example, for $n = 8$ transmit antennas, the maximum rate of CODs with square size is $1/2$ [4] while the COD obtained here has a rate of $5/8$. This phenomenon was observed in [2] in case of real orthogonal designs.

Note that for $n > 10$ transmit antennas, the block length of the obtained CODs is large. For example, for $n = 16$ transmit antennas, the block length is $p = 22\,880$. Thus, in this case, the resulting COD is more of theoretical interest.

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